Katz Centrality (directed graphs)
**Recall:**

<table>
<thead>
<tr>
<th>Quality: what makes a node important (central)</th>
<th>Mathematical Description</th>
<th>Appropriate Usage</th>
<th>Identification</th>
</tr>
</thead>
</table>
| Lots of one-hop connections from $v$          | The number of vertices that $v$ influences directly | Local influence matters Small diameter | Degree $\text{deg}(i)$  
| Lots of one-hop connections from $v$ relative to the size of the graph | The proportion of the vertices that $v$ influences directly | Local influence matters Small diameter | Degree centrality $C_i = \frac{\text{deg}(i)}{|V(G)|}$  
| Lots of one-hop connections to high centrality vertices | A weighted degree centrality based on the weight of the neighbors (instead of a weight of 1 as in degree centrality) | For example when the people you are connected to matter. | Eigenvector centrality (recursive formula): $C_i \propto \sum_j C_j$  

**Directed graphs?**
Katz Centrality

• Recall that the eigenvector centrality $x(t)$ is a weighted degree obtained from the leading eigenvector of $A$: $A x(t) = \lambda_1 x(t)$, so its entries are

$$x_i = \frac{1}{\lambda_1} \sum_j A_{ij} x_j$$

Thoughts on how to modify the above formula?

• Katz centrality: $x_i = \frac{1}{\lambda_1} \sum_j A_{ij} x_j + \beta$,

Where $\beta$ is a constant initial weight given to each vertex so that vertices with zero in degree (or out degree) are included in calculations.
Strongly connected (the $\beta$ part)

**Definition:** A directed graph $D = (V, E)$ is *strongly connected* if and only if, for each pair of nodes $u, v \in V$, there is a path from $u$ to $v$.

- The Web graph is not strongly connected since
  - there are pairs of nodes $u$ and $v$, there is no path from $u$ to $v$ *and* from $v$ to $u$.

Add a link from each page to every page and give each link a small transition probability controlled by a parameter $\beta$.

Katz centrality

After this augmentation, at a page, the random surfer has two options:

- He randomly chooses an out-link to follow \( A_{ij} \)
- He jumps to a random page \( \beta \)

• Katz centrality: 
  \[ x_i = \frac{1}{\lambda_1} \sum_j A_{ij} x_j + \beta, \]

Where \( \beta \) is a constant initial weight given to each vertex so that vertices with zero in degree (or out degree) are included in calculations.
An extension

• Katz centrality: \( x_i = \frac{1}{\lambda_1} \sum_j A_{ij} x_j + \beta \),

Where \( \beta_i \) is an initial weight given to vertex \( i \) as a mechanism to differentiate vertices using some quality not modeled by adjacencies. Vertices with zero in degree (or out degree) will be included in calculations.
Katz Centrality

Does

• Generalize the concept of eigenvector centrality to directed networks that are not strongly connected

Does not

• Control for the fact that a high centrality vertex imparts high centrality on those vertices “downstream,” or all those vertices reachable from that high centrality vertex
## Overview:

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<td>Local influence matters Small diameter</td>
<td>Normalized degree centrality ( C_i = \text{deg}(i) )</td>
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<td>A weighted degree centrality based on the out degree of the neighbors</td>
<td>Directed graphs that are not strongly connected</td>
<td>Katz centrality ( x_i = \frac{1}{\lambda_1} \sum_j A_{ij} x_j + \beta ), Where ( \beta ) is some initial weight</td>
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