MA4404 Complex Networks

Katz Centrality for directed graphs
Learning Outcomes

• Understand how Katz centrality is an extension of Eigenvector Centrality to directed graphs.
• Compute Katz centrality per node.
• Interpret the meaning of the values of Katz centrality.
## Recall: Centralities

<table>
<thead>
<tr>
<th>Quality: what makes a node important (central)</th>
<th>Mathematical Description</th>
<th>Appropriate Usage</th>
<th>Identification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lots of one-hop connections from $v$</td>
<td>The number of vertices that $v$ influences directly</td>
<td>Local influence matters Small diameter</td>
<td>Degree $\text{deg}(i)$</td>
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<tr>
<td>Lots of one-hop connections from $v$ relative to the size of the graph</td>
<td>The proportion of the vertices that $v$ influences directly</td>
<td>Local influence matters Small diameter</td>
<td>Degree centrality $C_i = \frac{\text{deg}(i)}{</td>
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<tr>
<td>Lots of one-hop connections to high centrality vertices</td>
<td>A weighted degree centrality based on the weight of the neighbors (instead of a weight of 1 as in degree centrality)</td>
<td>For example when the people you are connected to matter.</td>
<td>Eigenvector centrality (recursive formula): $C_i \propto \sum_j C_j$</td>
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**Recall: Strongly connected**

**Definition:** A directed graph $D = (V, E)$ is **strongly connected** if and only if, for each pair of nodes $u, v \in V$, there is a path from $u$ to $v$.

- The Web graph is not strongly connected since
  - there are pairs of nodes $u$ and $v$, there is no path from $u$ to $v$ and from $v$ to $u$.
  - This presents a challenge for nodes that have an in-degree of zero.

Add a link from each page to every page and give each link a small transition probability controlled by a parameter $\beta$.

Katz Centrality

• Recall that the eigenvector centrality $\mathbf{x}(t)$ is a weighted degree obtained from the leading eigenvector of $A$: $A \mathbf{x}(t) = \lambda_1 \mathbf{x}(t)$, so its entries are

$$x_i = \frac{1}{\lambda_1} \sum_j A_{ij} x_j$$

Thoughts on how to adapt the above formula for directed graphs?

• Katz centrality: $x_i = \frac{1}{\lambda_1} \sum_j A_{ij} x_j + \beta$,

Where $\beta$ is a constant initial weight given to each vertex so that vertices with zero in degree (or out degree) are included in calculations.

• After this augmentation, a random surfer on a particular webpage, has two options:
  ✓ He randomly chooses an out-link to follow ($A_{ij}$)
  ✓ He jumps to a random page ($\beta$)

Does $\beta$ have to be the same for each vertex?

• An extension: $\beta_i$ is an initial weight given to vertex $i$ as a mechanism to differentiate vertices using some quality not modeled by adjacencies. Vertices with zero in degree (or out degree) will be included in calculations.
Katz Centrality

Does

• Generalize the concept of eigenvector centrality to directed networks that are not strongly connected

Does not

• Control for the fact that a high centrality vertex imparts high centrality on those vertices “downstream,” or all those vertices reachable from that high centrality vertex → PageRank
## Updated Overview:

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<td>A weighted degree centrality based on the out degree of the neighbors</td>
<td>Directed graphs that are not strongly connected</td>
<td>Katz centrality ( x_i = \frac{1}{\lambda_1} \sum_j A_{ij} x_j + \beta ), Where ( \beta ) is some initial weight</td>
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