MA4404 Complex Networks

*Eigenvector Centrality*
Learning Outcomes

• Compute eigenvector centrality.
• Interpret the meaning of the values of eigenvector centrality.
• Explain why the eigenvector centrality is an extension of degree centrality.
Eigenvector Centrality
Recall:

<table>
<thead>
<tr>
<th>Quality: what makes a node important (central)</th>
<th>Mathematical Description</th>
<th>Appropriate Usage</th>
<th>Identification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lots of one-hop connections from $v$</td>
<td>The number of vertices that $v$ influences directly</td>
<td>Local influence matters Small diameter</td>
<td>Degree $\deg(i)$</td>
</tr>
<tr>
<td>Lots of one-hop connections from $v$ relative to the size of the graph</td>
<td>The proportion of the vertices that $v$ influences directly</td>
<td>Local influence matters Small diameter</td>
<td>Degree centrality $C_i = \frac{\deg(i)}{</td>
</tr>
<tr>
<td>Lots of one-hop connections to high centrality vertices</td>
<td>A weighted degree centrality based on the weight of the neighbors (instead of a weight of 1 as in degree centrality)</td>
<td>For example when the people you are connected to matter.</td>
<td><strong>HOW? Eigenvector centrality (recursive formula):</strong> $C_i \propto \sum_j C_j$</td>
</tr>
</tbody>
</table>
Eigenvector Centrality

• A generalization of the degree centrality: a weighted degree vector that depends on the centrality of its neighbors (rather than every neighbor having a fixed centrality of 1)

• How do we find it? By finding the largest eigenvalue and its associated eigenvector (leading eigenvector) of the adjacency matrix

• Let’s see why
Example 1 (Eigenvector centrality)

<table>
<thead>
<tr>
<th>Node $i$</th>
<th>Eigenvector centrality $C_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.5298987782873977</td>
</tr>
<tr>
<td>2</td>
<td>0.3577513877490464</td>
</tr>
<tr>
<td>3</td>
<td>0.5298987782873977</td>
</tr>
<tr>
<td>4</td>
<td>0.3577513877490464</td>
</tr>
<tr>
<td>5</td>
<td>0.4271328349194304</td>
</tr>
</tbody>
</table>

Notice that $\text{deg}(5) = \text{deg}(1) = \text{deg}(3)$. Why $C_5 > C_2$ and $C_5 > C_4$?
\[ x_i(t) = \sum_j A_{ij} x_j(t - 1) \]

with the centrality at time \( t=0 \) being \( x_j(0) = 1, \forall j \)
Eigenvector Centrality

• Define the centrality $x'_i$ of $i$ recursively in terms of the centrality of its neighbors

$$x'_i = \sum_{k \in N(i)} x_k \quad or \quad x'_i = \sum_j A_{ij} x_j$$

With initial vertex centrality $x_j = 1, \forall j$ (including $i$)—we’ll see why on next slide

• That is equivalent to:

$$x_i(t) = \sum_j A_{ij} x_j(t-1)$$

with the centrality at time $t=0$ being $x_j(0) = 1, \forall j$
In class: Eigenvector Centrality

Adjacency matrix $A$ for the graph to the right:

$$A = \begin{pmatrix}
0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}$$

Then the vector $x(t) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ gives a random surfer’s behavior.

Answer the following questions based on the information above
Q1: Find $x(1)$. What does it represent?

Answer: $x(1) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = (?)$
In class activity: Eigenvector Centrality

Q1: Find \( x(1) \). What does it represent?

\[
x(1) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 0 \\ 2 \end{pmatrix}
\]

The Degree Centrality vector
Q2: Find $x(2)$. What does it represent?

Answer: $x(2) = \begin{pmatrix}
0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 \\
\end{pmatrix} \begin{pmatrix}
3 \\
3 \\
3 \\
0 \\
3 \\
2 \\
\end{pmatrix} = (?)$
In class activity: Eigenvector Centrality

Q2: Find $x(2)$. What does it represent?

Answer: $x(2) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 3 \\ 0 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ 8 \\ 0 \\ 8 \\ 6 \end{pmatrix}$

A weighted Degree Centrality vector (distance 2 or less)
In class activity: Eigenvector Centrality

Q3: Find $x(3)$. What does it represent?

Answer: $x(3) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 9 \\ 9 \\ 8 \\ 0 \\ 8 \\ 6 \end{pmatrix} = (?)$
In class activity: Eigenvector Centrality

Q3: Find $x(3)$. What does it represent?

Answer: $x(3) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 9 \\ 9 \\ 8 \\ 0 \\ 8 \\ 6 \end{pmatrix} = \begin{pmatrix} 25 \\ 25 \\ 24 \\ 0 \\ 24 \\ 16 \end{pmatrix}$

A weighted Degree Centrality vector (distance 3 or less)
### In class: Eigenvector Centrality Results

<table>
<thead>
<tr>
<th>Node $i$</th>
<th>Eigenvector centrality $C_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.49122209552166</td>
</tr>
<tr>
<td>1</td>
<td>0.49122209552166</td>
</tr>
<tr>
<td>2</td>
<td>0.4557991200411896</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0.4557991200411896,</td>
</tr>
<tr>
<td>5</td>
<td>0.31921157573304415</td>
</tr>
</tbody>
</table>
The derivation of eigenvector centrality

\[ x(t) = A(A \ldots (A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix})) = A^t x(0), \quad t > 0 \]
Discussion: What did you notice?

• What is $x(3)$?

Answer: $x(3) = A(A(A(1))) = A^3 x(0)$

$x(3)$ depends on the centrality of its distance 3 or less neighbors

• What is $x(t)$?

Answer: $x(t) = A(A ... (A(1))) = A^t x(0), t > 0$

$x(t)$ depends on the centrality of its distance $t$ or less neighbors
Eigenvector Centrality Derivation

• We can consolidate the eigenvector centralities of all the nodes in a recursive formula with vectors:

\[ x(t) = A \cdot x(t - 1) \]

with the centrality at time \( t=0 \) being \( x(0) = 1 \) (as a vector)

• Then, we solve: \( x(t) = A^t \cdot x(0) \), with \( x(0) = 1 \)
• Let \( \nu_k \) be the eigenvectors of the adjacency matrix \( A \)
• Let \( \lambda_1 \) be the largest eigenvalue.
• Let \( x(0) = \sum_k c_k \nu_k \), be a linear combination of \( \nu_k \) (eigenvectors are orthogonal since \( A \) is real and symmetric)
Eigenvector Centrality Derivation

Facts from previous page:
• \( \mathbf{x}(t) = A^t \cdot \mathbf{x}(0) \), with \( \mathbf{x}(0) = 1 \)
• \( \mathbf{v}_k \) are the eigenvectors of the adjacency matrix \( A \)
• \( \mathbf{x}(0) = \sum_k c_k \mathbf{v}_k \) is a linear combination of \( \mathbf{v}_k \)
• \( \lambda_1 \) be the largest eigenvalue.

Then
\[
\mathbf{x}(t) = A^t \cdot \mathbf{x}(0) = A^t \sum_k c_k \mathbf{v}_k = \sum_k c_k \lambda_k^t \mathbf{v}_k = \\
= \lambda_1^t \sum_k c_k \frac{\lambda_k^t}{\lambda_1^t} \mathbf{v}_k = \lambda_1^t \left( c_1 \frac{\lambda_1^t}{\lambda_1^t} \mathbf{v}_1 + c_2 \frac{\lambda_2^t}{\lambda_1^t} \mathbf{v}_2 + \ldots + c_n \frac{\lambda_n^t}{\lambda_1^t} \mathbf{v}_n \right) \to \lambda_1^t c_1 \mathbf{v}_1
\]

since \( \frac{\lambda_k^t}{\lambda_1^t} \to 0 \) as \( t \to \infty \) (as you repeat the process)
Eigenvector Centrality

• Thus the eigenvector centrality is
  \[ x(t) = \lambda_1^t c_1 v_1 \]
  where \( v_1 \) is the eigenvector corresponding to the largest eigenvalue \( \lambda_1 \)

• So the eigenvector centrality (as a vector), \( x(t) \), is a multiple of the eigenvector \( v_1 \), i.e. \( x(t) \) is an eigenvector of \( A \).
  \[ A x(t) = \lambda_1^t x(t) \]

• Meaning that the eigenvector centrality of each node is given by the entries of the leading eigenvector (the one corresponding to the largest eigenvalue \( \lambda = \lambda_1^t \))
Is it well defined?

• That is:
  • Is the eigenvector guaranteed to exist?
  • Is the eigenvector unique?
  • Is the eigenvalue unique?
  • Can we have negative entries in the eigenvector?

• We say that a matrix/vector is positive if all of its entries are positive

• Perron-Frobenius theorem: A real square matrix with positive entries has a \textit{unique largest real eigenvalue} and that the \textit{corresponding eigenvector has strictly positive components}

• Perron-Frobenius theorem applies to positive matrices (but it gives similar information for nonnegative ones)
Perron-Frobenius theorem for nonnegative symmetric (0,1)-matrices

Let $A \in \mathbb{R}^{n \times n}$ be symmetric (0,1)-nonnegative, then

- there is a unique maximal eigenvalue $\lambda_1$ of the matrix $A$ (for any other eigenvalue $\lambda$, we have $\lambda < \lambda_1$, with the possibility of $|\lambda| = \lambda_1$ for nonnegative matrices)

- $\lambda_1$ is real, simple (i.e., has multiplicity one), and positive (trace is zero so some are positive and some negative),

- the associated eigenvector is nonnegative (and there are no other nonnegative ones since all eigenvectors are orthogonal)

If you have not seen this and its proof in linear algebra, see a proof on pages 346-347 of Newman’s textbook.
Consider the vectors computed:

\[
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 0 \\ 3 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 9 \\ 9 \\ 0 \\ 8 \\ 8 \\ 6 \end{pmatrix}, \begin{pmatrix} 25 \\ 25 \\ 24 \\ 24 \\ 16 \end{pmatrix}
\]

- In finding the eigenvector, these vectors get normalized as they are computed using the power method from Linear Algebra, and eventually converge to a normalized eigenvector as well.
- Note that \( \frac{x_i(2)}{x_i(1)} \neq \frac{x_i(3)}{x_i(2)} \), where \( x_i \) is the \( i^{th} \) entry, however, the ratios will converge to \( \lambda_1 \).
Conclusion: Eigenvector Centrality

• Therefore:
  • it is a generalized degree centrality (takes into consideration the global network)
  • It is extremely useful, one of the most common ones used for non-oriented networks
  • $C_i \propto \sum_j C_j$ or $C_i = \lambda^{-1} \sum_j A_{ij} C_j$ or $C_i = \sum_{ij \in E(G)} C_j$

• Why is Eigenvector Centrality not commonly used for directed graphs?
  • Adjacency matrix is asymmetric...use left or right leading eigenvector?
  • Choose right leading eigenvector...importance bestowed by vertices pointing toward you (same problem with left).
    • Any vertex within degree zero has centrality value zero and “passes” that value to all vertices to which it points.

• The fix: Katz centrality
Extra example 2 (Adjacency matrix, eigenvector centrality and the graph)

0: 0.08448651593556764,
1: 0.1928608426462633,
2: 0.3011603786470362,
3: 0.17530527234882679,
4: 0.40835121533077895,
5: 0.2865100597893966,
6: 0.2791290343288953,
7: 0.1931920790704947,
8: 0.24881035953707603,
9: 0.13868390351302598,
10: 0.336067959653752,
11: 0.16407815738375933,
12: 0.33838887484747293,
13: 0.2871391639624871,
14: 0.22848023925633135
Extra example 2 (Eigenvector centrality)

\[ C_2 = 0.3011603786470362 \]

Adjacent to vertices of small degree

\[ C_4 = 0.40835121533077895 \]

Adjacent to vertices of large degree