Choose exactly 3 of the following problems. Only the first 3 problems will be graded if more than 3 are turned in.

1. Let $S$ be the set of all polynomials of degree 7 such that the constant term is 0 (i.e. $p(0) = 0$ for all such polynomials).
   (a) Is $S$ a subspace?
   (b) If so, what is its dimension?
   (c) What is a basis for $S$?
   (d) Give an example of 3 linearly independent vectors in $S$.

2. How many solutions does $Ax = b$ have if:
   (a) $b \notin$ column space of $A$
   (b) $b = 0$ and $rankA < $ number of columns of $A$
   (c) $b \in$ column space of $A$ and $rankA < $ number of columns of $A$
   (d) If $A$ is $10 \times 10$ and $rankA = 7$, what is the dim$N(A)$?
   (e) $A$ is an $n \times n$ positive definite and $b \in$ column space of $A$

3. Do all of the following:
   (a) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \left(\begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix} \right)$, $\forall x \in \mathbb{R}^2$. Find the matrix $K$ representing $L$ with respect to the bases $\{e_1, e_2\}$ and $\{b_1, b_2\}$, where $b_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$.
   (b) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = -x_2 b_1 - x_1 b_2$ (or $L(x) = -x_2 b_1 + -x_1 b_2$), $\forall x \in \mathbb{R}^2$, where $b_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$. Find the matrix $D$ representing $L$ with respect to the ordered bases $\{e_1, e_2\}$ and $\{b_1, b_2\}$.
   (c) find the Kernel of the linear transformation in (a)
   (d) find Image$(S)$, where $S = \left\{ \begin{bmatrix} 0 \\ x \end{bmatrix} : x \in \mathbb{R} \right\}$ and $L$ is defined in (a) above
4. Let \( f(x, y, z) = x^2 + 3y^2 - z^2 - 4xy + 4yz \).
   (a) find the matrix \( A \) corresponding to \( f \)
   (b) find the \( N(A) \)
   (c) find a basis for \( R(A) \)
   (d) what is the dimension of \( N(ATA) \)?
   (e) is \( A \) positive definite?
   (f) is \( A \) diagonalizable?

5. Let \( f, g \) be two functions in \( C[-1, 1] \), and define an inner product on \( C[-1, 1] \) by \( \langle f, g \rangle = \int_{-1}^{1} fg \, dx \).
   (a) find \( \langle e^{-x}, e^x \rangle \)
   (b) find \( ||\cos(\frac{x}{2})|| \)
   (c) find two functions that are orthogonal in the set \( \{1, e^{-x}, e^x, \cos(\frac{x}{2}), x\} \). Are the two functions you chose orthonormal? Why?
   (d) Give an example of a \( 4 \times 4 \) orthogonal matrix and explain why it is orthogonal (this part is not related to the inner product defined above, use the standard inner product in \( \mathbb{R}^4 \))

6. Let \( f(x, y, z) = x^3 + xyz + y^2 - 3x \).
   (a) Find the Hessian \( H \) of \( f \) at \( x_0 = (1, 0, 0)^T \) and, along the way, verify that \( (1, 0, 0) \) is a stationary point for \( f \).
   (b) Is the matrix from (a) positive definite? Explain.
   (c) Does \( f \) have a maximum, a minimum, or a saddle point at \( (1, 0, 0) \)?
   (d) find the Frobenius norm of \( H \) at \( (1, 0, 0) \), i.e. find \( ||H||_F((1,0,0)) \)

7. Let \( A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 2 & 4 \end{bmatrix} \).
   (a) Find the singular value decomposition of \( A \).
   (b) Produce orthonormal bases for \( R(A) \), \( R(ATA) \), \( N(A) \), and \( N(ATA) \).
   (c) Construct the best rank-one approximation of \( A \).

8. Let \( f(x, y) = 2x^3 + x^2 + 2y^2 - 4xy + 2 \).
   (a) Verify that \( (0, 0) \) and \( (1/3, 1/3) \) are stationary points for \( f \).
   (b) Construct the Hessian of \( f \) for each stationary point.
   (c) Does \( f \) have a maximum, a minimum, or a saddle point at \( (0,0) \)? At \( (1/3,1/3) \)?

9. Suppose that \( A \) is a real \( 5 \times 4 \) matrix with singular value decomposition given by

\[
A = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \\ v_4^T \end{bmatrix}.
\]

Assume that \( \sigma_1 \geq \sigma_2 \geq \sigma_3 > 0 \).
(a) What is the rank of $A$?
(b) How is $v_1$ computed?
(c) How is $u_1$ computed?
(d) Show that $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \sigma_3 u_3 v_3^T$, and that we can therefore obtain a more compact SVD for $A$ by discarding columns 4 and 5 of $U$, rows 4 and 5 and column 4 of $\Sigma$, and column 4 of $V$.
(e) What matrix $B$ is the best rank-one approximation to $A$?

10. Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \\ 0 & 0 \end{bmatrix}$. In the singular value decomposition of $A$, we have $\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$, and can use $V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

(a) Find $U$ such that $A = U\Sigma V^T$.
(b) From the columns of $U$ and $V$, produce orthonormal bases for $R(A)$, $R(A^T)$, $N(A)$, and $N(A^T)$.
(c) Use the singular value decomposition of $A$ to construct the best (in terms of $\| \cdot \|_F$) rank-one approximation of $A$.
(d) Based on (c), write $A$ as a sum $A = \sigma_1 A_1 + \sigma_2 A_2$, where $A_1$ and $A_2$ are rank 1 matrices.

**Solution:**

(a) We easily find $u_1 = \frac{1}{4} A v_1 = (1/\sqrt{2}, 1/\sqrt{2}, 0)^T$ and $u_2 = \frac{1}{2} A v_2 = (-1/\sqrt{2}, 1/\sqrt{2}, 0)^T$. We may take $u_3 = (0, 0, 1)^T$.
(b) The bases in question are
   i. $\{u_1, u_2\}$, for $R(A)$
   ii. $\{u_3\}$, for $N(A^T)$
   iii. $\{v_1, v_2\}$, for $R(A^T)$
   iv. $A$ has the trivial nullspace.
(c) To obtain the best rank-one approximation, say $A'$, we zero the smaller singular value, obtaining

$$A' = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 0 & 0 \end{bmatrix}.$$ 
(d) From (c), we can see that $\sigma_2 A_2 = A - A' = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}$, so the expression we want is

$$A = \sigma_1 A_1 + \sigma_2 A_2 = 4 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \\ 0 & 0 \end{bmatrix}.$$ 

11. Let $L$ be the operator on $P^3$ defined by $L(p(x)) = p'(x) + x^2 p''(x)$.

(a) Describe the kernel and the range of $L$. 

(b) Construct the matrix representation $A$ of $L$ with respect to the basis $[1, x, x^2]$.  
(c) Construct the matrix representation $B$ of $L$ with respect to the basis $[1, x, x^2]$.  
(d) Find the matrix $S$ such that $B = S^{-1}AS$. (To verify, show that $SB = AS$.)  

Solution:  

(a) By applying $L$ to an arbitrary element $p(x) = ax^2 + bx + c$, we find $L(p(x)) = 2ax^2 + 2ax + b = 2a(x^2 + x) + b$. It follows that the kernel of $L$ is $P_1$ and the range is $Span(x^2 + x, 1)$.  

(b) The matrix representation of $L$ with respect to $[1, x, x^2]$ is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.  

(c) The matrix representation of $L$ with respect to $[1, x, x^2 + x]$ is $B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.  

(d) The matrix $S$ such that $B = S^{-1}AS$ is $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Verification is straightforward.