6.6 Quadratic Forms

1. A quadratic equation in two variables $x$ and $y$ is an equation of the form

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

which is equivalent to

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + f = 0.$$ 

2. The graph of the above equation is called a conic section.

3. Standard form for conics (2 variables):
   
   (a) Circle: $x^2 + y^2 = r^2$ with radius $r \neq 0$
   
   (b) Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with vertices $(\pm a, 0)$ and $(0, \pm b)$
   
   (c) Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with vertices $(\pm a, 0)$; or $\frac{y^2}{b^2} - \frac{x^2}{ba^2} = 1$ with vertices $(0, \pm b)$ $a \neq 0, b \neq 0$
   
   (d) Parabola: $x^2 = ay$ (or $y^2 = ax$, $a \neq 0$) or $x^2 = ay$ (or $x = ay^2$, $a \neq 0$)

4. If the coefficient of $xy$ is nonzero, then the conic had a rotation.

5. See graphs on page 353.

6. If a quadratic is not in standard form, then the conic had a translation along the $x$-axis (if there is an $x$ term along with the $x^2$ term) or $y$-axis (if there is a $y$ term along with the $y^2$ term). A combination of the 2 is also possible on a conic.

7. The 3-variable quadratics are called quadratic surfaces (see page 357).

8. Principal Axes Theorem: If $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix, then there is a change of variable $u = Q^T x$ such that $x^T A x = u^T D u$, where $D$ is a diagonal matrix.

9. Finding max/min, both local and global for quadratics:

   If $F(x) \in \mathbb{R}^n$ is a real-valued function such that $\partial F(x_0) = 0$ (i.e. all partial derivatives exist and they are zero), then $x_0$ is a stationary point (or critical point).

10. If $F(x) = x^T A x$ and $A$ is nonsingular, then $x_0 = (0, 0)$ is the only solution, and so it is a global max, min or saddle point. Particularly, if we only consider the pure quadratics $ax^2 + 2bxy + cy^2$ (or its variant in 3 or more variables) then

$$x_0 = (0, 0)$$

is the only stationary point $\iff$ $A$ is nonsingular $\iff$ the eigenvalues are nonzero.

Why? See next page with positive/negative definite.
11. note that if \( \lambda \) is an eigenvalue of \( A \), then
\[
x^T A x = x^T \lambda x = \lambda x^T x = \lambda ||x||^2.
\]
And so
\[
x^T A x > 0 \iff \text{its eigenvalues are all positive}
\]
\[
x^T A x < 0 \iff \text{its eigenvalues are all negative}
\]
12. a quadratic \( F(x) = x^T A x \) is \textit{definite} if \( x^T A x \) doesn’t change sign \( \forall x \in \mathbb{R}^n \). If so, then \( F(x) \) (and also \( A \)) can be
(a) \textit{positive definite} if \( x^T A x > 0, \forall x \in \mathbb{R}^n \), i.e. \( \lambda_i > 0, \forall i \)
(b) \textit{negative definite} if \( x^T A x < 0, \forall x \in \mathbb{R}^n \), i.e. \( \lambda_i < 0, \forall i \)
13. a quadratic \( F(x) = x^T A x \) is \textit{semidefinite} if \( x^T A x \) equals zero for some values of \( x \), and it has the same sign otherwise. If so, then \( F(x) \) (and also \( A \)) can be
(a) \textit{positive semidefinite} if \( x^T A x \geq 0, \forall x \in \mathbb{R}^n \), i.e. \( \lambda_i \geq 0, \forall i \)
(b) \textit{negative semidefinite} if \( x^T A x \leq 0, \forall x \in \mathbb{R}^n \), i.e. \( \lambda_i \leq 0, \forall i \)
14. a quadratic \( F(x) = x^T A x \) (and also \( A \)) is \textit{indefinite} if \( x^T A x \) changes sign for different values of \( x \in \mathbb{R}^n \) (it may but it doesn’t have to have a zero eigenvalue).
15. Now, how do we know if \( x_0 = (0,0) \) is a maximum, minimum or saddle point for a quadratic in \textit{standard form} (i.e. \( x^T A x = 0 \))? The same way we checked in \( \mathbb{R} \):
(a) if \( A \) is \textit{positive definite} (i.e. if \( x^T A x > 0, \forall x \neq x_0 \)), then \( x_0 \) is a \textbf{min}
(b) if \( A \) is \textit{negative definite} (i.e. if \( x^T A x < 0, \forall x \neq x_0 \)), then \( x_0 \) is a \textbf{max}
(c) if \( x^T A x \) changes sign and no zero eigenvalues, then \( x_0 \) is a \textbf{saddle point}
(d) if some eigenvalues are 0, then test is inconclusive
16. let \( F \) be \textit{any smooth function} around \( x_0 \), where \( x_0 \) is a stationary point (non necessarily zero) for \( F \). We can locally approximate \( F \) by a quadratic (using Taylor series), and determine if \( x_0 \) is a local min/max/saddle point \( (F \) may have more than one min/max/saddle point, so we check each \( x_0 \) at the time). We define \( H(x_0) \), the matrix \( (h_{ij}) \), where \( h_{i,j} = \frac{\partial^2 F}{\partial x_i \partial x_j} (x_0) \), to be the \textit{Hessian} of \( F \) at \( x_0 \). Then
(a) if \( H(x_0) \) is \textit{positive definite}, then \( x_0 \) is a min
(b) if \( H(x_0) \) is \textit{negative definite}, then \( x_0 \) is a max
(c) if \( H(x_0) \) changes sign, then \( x_0 \) is a saddle point
(d) if \( H(x_0) \) is \textit{semidefinite}, then test is inconclusive