4.1 Definitions and Examples

1. a linear transformation \( L : V \to W \) is a function such that

\[
L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2),
\]

where \( \alpha, \beta \) are constants, \( v_1, v_2 \) are vectors in \( V \), and \( L(v_1), L(v_2) \) are vectors in \( W \).

2. alternatively, one can define a linear transformation \( L : V \to W \) to be a function such that

\[
L(v_1 + v_2) = L(v_1) + L(v_2) \quad \text{and} \quad L(\alpha v_1) = \alpha L(v_1),
\]

where \( \alpha \) is a constant, \( v_1, v_2 \) are vectors in \( V \), and \( L(v_1), L(v_2) \) are vectors in \( W \).

3. if \( V = W \), then the linear transformation \( L \) is called a linear operator.

4. particularly, a linear transformation maps the zero vector of \( V \) to the zero vector on \( W \): \( L(0_V) = 0_W \) (let \( \alpha = \beta = 0 \) in the original definition to see that it must hold true)

5. similarly a linear transformation maps the negative of a vector of \( V \) to the negative of its image in \( W \): \( L(-v_V) = -v_W \) (let \( \alpha = -1, \beta = 0 \) in the original definition to see that it must hold true)

6. for a linear transformation \( L : V \to W \), we define

(a) the kernel of \( L \) to be

\[
\text{Ker}(L) = \{ v \in V : L(v) = 0_W \},
\]

(b) the image of a subspace \( S \subseteq V \) is

\[
L(S) = \{ w \in W : L(v) = w, \exists v \in S \}
\]

(c) the range of \( L \) is the image of \( V \) itself.

7. both the kernel and the image are subspaces

8. each linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) can be written as a matrix multiple of the input: \( L(v) = Av \) (we’ll do this in section 2).