CH3: VECTOR SPACES

3.6 Row Space and Column Space

1. row vectors/column vectors are the rows/columns of an \( m \times n \) matrix \( A \)

2. the rows will span a subspace of \( \mathbb{R}^{1\times n} \), which is called row space: rowS(\( A \)) or \( RS(\( A \)) \)

3. the columns will span a subspace of \( \mathbb{R}^{m} \), which is called column space: colS(\( A \)) or \( CS(\( A \)) \)

4. the subspace called column space, is a subspace of \( \mathbb{R}^{m} \), which may happen to be exactly \( \mathbb{R}^{m} \), or it could be a proper subspace. Similarly for column space.

5. if two matrices are row equivalent (for example one is obtained from the other using Row Echelon reduction) then they have the same row space (but generally not the same column space). To obtain a basis for the rowS(\( A \)), choose the pivotal rows of \( U \) (not of \( A \)).

6. That is, reduce \( A \) to Row Echelon Form \( U \), and use the columns of \( A \) (but not of \( U \)) corresponding to the pivots in \( U \) to obtain a basis for colS(\( A \)).

7. the rank of the matrix \( A \) is the dimension of the row space (i.e. the number of rows that have nonzero pivots)

8. the dimension(rowS(\( A \))) = dimension(colS(\( A \))) = the number of pivots in \( U \)

9. **Consistency Theorem** for linear systems: Let \( A \) be an \( m \times n \) matrix.

   (a) \( Ax = b \) is consistent \( \iff \) \( b \in \text{colS} (A) \)

   (b) \( Ax = b \) is consistent \( \forall b \in \mathbb{R}^m \iff \text{colS} (A) = \mathbb{R}^m \) \( (n \geq m) \)

   (c) for each \( b \), \( Ax = b \) has at most one solution (i.e. no solutions or exactly one solution) \( \iff \) columns of \( A \) are linearly independent \( (n \leq m) \)

10. Particularly, the Consistency Theorem implies the following:

   (a) If \( b \notin \text{colS} (A) \), then \( Ax = b \) has no solutions (system is inconsistent).

   (b) If \( b \in \text{colS} (A) \), then \( Ax = b \) has solutions with

      i. if the column vectors of \( A \) are lin. indep, then \( Ax = b \) has one solution

      ii. if the column vectors of \( A \) are not lin. indep, then \( Ax = b \) has infinitely many solution

11. \( A \in \mathbb{R}^{m\times n} \) is nonsingular \( \iff \) \( \mathbb{R}^{m} \) has as a basis the columns of \( A \) (i.e. the span of the columns of \( A \) is \( \mathbb{R}^{m} \))

12. same dimension of a space does not imply the spaces are the same: \( \mathbb{R}^{1\times n} \neq \mathbb{R}^{n\times 1} \)
13. Here is the connection between $N(A)$ and $\text{rank}(A)$: **Rank Nullity Theorem**:

$$\text{rank}(A) + \text{nullity}(A) = n,$$

where $\text{rank}(A) =$ the dimension of the row space (it is also the dimension of the column space, and so it is the number of lead or also called dependent variables, i.e. nonzero pivots),

$\text{nullity}(A) =$ the dimension of $N(A)$ (i.e. the number of free variables), and

$n =$ number of columns.