3.5 Change of Basis

1. changing from one coordinate system to another can make a problem easier to solve (See Application 1 that shows how one would know that the states stabilize after a period of time.)

2. Changing coordinates in $\mathbb{R}^2$: the standard basis is $[e_1 = (1,0)^T, e_2 = (0,1)^T]$, so every vector $(a, b)^T = ae_1 + be_2$, for scalars $a, b \in \mathbb{R}$. Now, the same vector $(a, b)^T$ has a unique representation in a different basis, say $[x, y]$, as $(a, b)^T = \alpha x + \beta y$, for scalars $\alpha, \beta \in \mathbb{R}$. We then refer to $\alpha$ and $\beta$ as the coordinates of the vector $(a, b)^T$ with respect to the basis $[x, y]$. The coordinates specify how many copies of $x$ and $y$ need to be used so that the sum is exactly the vector $(a, b)^T$ in the new coordinate system.

3. if we use the basis $[y, x]$ then the coordinates of $(a, b)^T$ are not $\alpha$ and $\beta$ anymore.

4. once we have decided what basis to work in, what are the coordinates of the given vector in the new basis? There are three cases:

   (a) Given $x = c_1 u_1 + c_2 u_2$, find the coordinates with respect to the standard basis $[e_1, e_2]$.
   
   - obtain the transition matrix $U = [u_1 \ u_2]$
   - every vector $x$ can be written in the standard basis as $Uc$

   (b) Given $x = x_1 e_1 + x_2 e_2$, find its coordinates with respect to the basis $[u_1, u_2]$.
   
   - find $U = [u_1 \ u_2]$
   - find $U^{-1}$ so that $c = U^{-1}x$
   - every vector $x$ can be written in the new basis $[u_1, u_2]$ as $U^{-1}x$

   (c) Given $x = c_1 v_1 + c_2 v_2$ in the basis $[v_1, v_2]$, find the coordinates of $x$ with respect to the new basis $[u_1, u_2]$, say $x = d_1 u_1 + d_2 u_2$
   
   - find $U = [u_1 \ u_2]$ and $V = [v_1 \ v_2]$
   - find $U^{-1}$ so that $d = U^{-1}Vc$
   - every vector $x$ can be written in the new basis $[u_1, u_2]$ as $U^{-1}Vc$

5. the above procedure applies to any dimension, not just 2.