3.4 Basis and Dimension

1. A set of vectors \( \{v_1, v_2, \ldots, v_k\} \) forms a basis for a vector space \( V \) iff:
   (a) \( v_1, v_2, \ldots, v_k \) are linearly independent, and
   (b) \( v_1, v_2, \ldots, v_k \) span \( V \).
   We then call \( k \) the dimension of the basis (see Cor 3.4.2.)

2. In order to show that a particular property holds for a vector space, one would have to show it for every vector in the vector space, or better: show that the property holds true for the elements of the basis. Since every vector in the vector space can be written as a linear combination of the basis elements, the property then holds true for every element of that vector space.

3. A basis is not unique.

4. If a basis for a vector space has \( k \) vectors, then any collection of \( m \) \((m > k)\) vectors is linearly dependent (see #3(b) page 150.) However not every collection of at most \( k \) vectors is linearly independent (see #4 page 150.)

5. A vector \( v \) spans a one-dimensional sub-space (in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) for example this subspace would be a line through the origin). Note that the origin belongs to the subspace since \( 0 \) belongs to every subspace.

6. Two linearly independent vectors \( v, u \) span a two-dimensional sub-space (in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) for example this subspace would be a plane through the origin). If the vectors are linearly dependent, then they span a one-dimensional space.

7. Three linearly independent vectors \( u, v, w \) span a three-dimensional sub-space (in \( \mathbb{R}^3 \) for example this subspace is \( \mathbb{R}^3 \) itself).

8. Generally it is tricky to prove that a set of vectors is a minimal spanning set. See #5(c) page 150 for a good example. And so the best case scenario is this: Let \( V \) be a space of dimension \( k > 0 \) (like \( \mathbb{R}^k \)), and let \( S \) be a set of \( k \) vectors. Then

   \[ \text{the vectors in } S \text{ are linearly independent } \iff \text{ } S \text{ spans } V. \]

And so, if we can show that a set of the same cardinality as the dimension of the vector space is lin. indep., then it is a maximal lin. indep. set. Similarly, if we can show that a set of the same cardinality as the dimension of the vector space spans \( V \), then it is a minimal spanning set. That is to say that the \( k \times k \) matrix whose columns are the \( k \) vectors must have \( \det \neq 0 \). If the set of vectors has more than \( k \) vectors, then find \( k \) of them whose matrix will have a nonzero determinant, and these \( k \) vectors will form a basis (like #3(a)). If the set of
vectors has fewer than \( k \) vectors, then find the largest submatrix whose matrix that will have a nonzero determinant, and the columns in this matrix tells you which vectors will form a basis (like \( \#8(a) \)).

9. Moreover, no set with less than \( k \) vectors can span \( V \), and no set with more than \( k \) vectors is linearly independent. However, each of them can be adjusted to a set with linearly indep. vectors or that spans \( V \) (See Thm 3.4.4).

10. a standard basis for a vector space is the most natural basis for that space:

(i) for \( \mathbb{R}^n \) this is \( \{ e_1, e_2, \ldots, e_n \} \), where \( e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \), where 1 is in the \( i^{th} \) row.

For \( \mathbb{R}^4 \), we have \( e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \).

(ii) for the space \( \mathbb{R}^{2 \times 2} \) this is \( \{ E_{11}, E_{12}, E_{21}, E_{22} \} \) given in example 2. page 146: \( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \).

(iii) for the space of polynomials \( P_n \) this is \( \{ 1, x, x^2, x^3, \ldots, x^{n-1} \} \),

(iv) the space of continuous or differentiable functions cannot be finitely generated as the ones above.