1. (10 points) Suppose that the vertex set of a graph $G$ is the set $V(G) = \{x_1, x_2, x_3, \ldots, x_{10,000}\}$. Two vertices $x_i$ and $x_j$ are adjacent if $i + j$ is odd. What is the graph $G$? (identify the class and/or name of the graph).

**Solution** The graph is the complete bipartite graph $K_{5000,5000}$, whose bipartite sets are $V_1 = \{x_{2k-1} : 1 \leq k \leq 5000\}$ and $V_2 = \{x_{2k} : 1 \leq k \leq 5000\}$, i.e. $V_1$ contains all the vertices with odd subscript, and $V_2$ contains all vertices with even subscript.

2. (10 points) Consider the three graphs $F$, $G$, $H$ below. Find two graphs in $\{F, G, H\}$ that are isomorphic, and two that are NOT isomorphic [explanations required].

**Solution:** $F \cong H$

$G \not\cong F$ since (each one of the reasoning below would suffice):

(1) $G$ has the vertices of degree 3 on the 5-cycle and $F$ doesn’t, or

(2) $F$ has three vertices of degree two that form a path on three vertices, and $G$ doesn’t, or

(3) $F$ has a cycle of length 6 and $H$ doesn’t.

3. (10 points) Find $W_7^c$, the complement of the wheel $W_7$. Is $W_7^c$ regular? Is it Eulerian?

**Solution:** $W_7^c$ is not regular since the degree sequence is 4, 4, 4, 4, 4, 4, 0. It is also not Eulerian since it is disconnected so no Eulerian circuit will traverse the center vertex of $W_7$. 
4. (10 points) Give an example of a graph that has an Eulerian circuit but not a Hamiltonian circuit.

Solution:

5. (10 points) Let $G_n$ be the complement of the cycle $C_n$ ($n \geq 3$). What is the degree sequence of $G_n$ ($n \geq 3$)?

Solution: Since each vertex in $C_n$ has degree 2, it follows that every vertex in $G_n$ has degree $(n - 1) - 2$. Thus the degree sequence is $n - 3, n - 3, \ldots, n - 3$.

6. (10 points) Let $G_n$ be the complement of the cycle $C_n$ ($n \geq 3$). Determine with justifications, which graphs $G_n$ ($n \geq 3$) have an Eulerian circuit.

Solution: Since $G_3$ and $G_4$ are not even connected, they will not have an Eulerian circuit. For $G_n, n \geq 5$, note that deg $v = n - 3$, which is even if $n$ is odd and $G$ connected. And so $G_n$ has an Eulerian circuit for odd values of $n \geq 5$. 
7. (10 points) Let $R$ be a relation defined on the positive integers by $xRy$ if $x|y$. Prove that $R$ is transitive.

Solution: Let $xRy$ and $yRz$. Then $x|y$ and $y|z$, so $y = xk$, $\exists k \in \mathbb{Z}^+$ and $z = y\ell$, $\exists \ell \in \mathbb{Z}^+$. Therefore $z = xk\ell$, $\exists k\ell \in \mathbb{Z}^+$, and so $xRz$.

8. (10 points) Let $S$ be the relation on a set of 4 elements given by the matrix below.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$


Solution: Yes, it is reflexive since the matrix has only 1s on the diagonal. Yes, the relation is symmetric since $A = A^T$.

9. (10 points) Find the equivalence classes of the equivalence relation $T$ on $\mathbb{Z}$, where $(x, y) \in T$ if $x + y$ is even, $\forall x, y \in \mathbb{Z}$. (You do not need to prove that it is an equivalence relation.)

Solution: There are two equivalence classes, $[0] = \{2k : k \in \mathbb{Z}\}$ and $[1] = \{2k+1 : k \in \mathbb{Z}\}$.

10. (10 points) CHROMATIC NUMBER WILL BE COVERED ON MONDAY’S LECTURE (The chromatic number is the minimum number of colors assigned to the vertices so that adjacent vertices have different colors): Find (and prove) the chromatic number for the graph below.
Solution: The $\chi(G) \leq 4$ since it is planar (by the 4 color Thm.), or you could find a coloring using 4 colors. Then also $\chi(G) \geq 4$ since it contains a wheel $W_5$ whose chromatic number is 4. Therefore $\chi(G) = 4$.

**Extra Credit (5 points)** Find a self-complementary graph (other than the path on four vertices from problem 50 Section 9.4).

Solution: $C_5$

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MA 3025  Sample Take Home Exam #3

December 5, 2011  Name _________________________________

Please use you notes and books only, and organize your work nicely. You may replace one problem on the exam with the following:

Let $G_1, G_2,$ and $G_3$ be pairwise disjoint connected regular graphs and let $G = G_1 + G_2 + G_3$ be the graph obtained from $G_1, G_2,$ and $G_3$ by adding edges between every two vertices belonging to two of $G_1, G_2,$ and $G_3$ (that is, there is an edge between each vertex of $G_1$ and each vertex of $G_2,$ there is an edge between each vertex of $G_1$ and each vertex of $G_3,$ and there is an edge between each vertex of $G_2$ and each vertex of $G_3$). Recall that $\overline{G_1}$ is the complement of $G_1,$ that has the same vertices of $G_1,$ and all the edges that $G_1$ is missing to be a complete graph. Prove that if $G_1$ and $\overline{G_1}$ are eulerian, but $G_2$ and $G_3$ are not eulerian, then $G$ is eulerian.

**Proof.** Let $G_i$ be $r_i$ - regular of order $n_i$ ($i = 1, 2, 3$). Since $G_1$ is eulerian, $r_1$ is even. Since $\overline{G_1}$ is eulerian, $n_1 - r_1 - 1$ is even. Thus $n_1$ is odd. Since $G_2$ is not eulerian, $r_2$ is odd and so $n_2$ is even (a graph must have an even number of odd degree vertices). Similarly, $r_3$ is odd and so $n_3$ even. Hence:

1. every vertex of $G$ in $G_1$ has degree $r_1 + n_2 + n_3$, which is even,
2. every vertex of $G$ in $G_2$ has degree $r_2 + n_1 + n_3$, which is even,
3. every vertex of $G$ in $G_3$ has degree $r_3 + n_1 + n_2$, which is even.

Since $G$ is connected and every vertex has even degree, $G$ is eulerian. $F \cong H \cong K_{3,3}$