Take home alternative DUE by Thursday 3pm.

(in the main office -Spanagel 250- or you can slide it under my office door
-Spanagel 260).

Please use you notes and books only, and organize your work nicely. You may replace one problem on the exam with the following:

(25 points) Prove that for all \( n \geq 1, \)

\[
\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \ldots + \frac{n}{2^n} = \frac{2^{n+1} - 2 - n}{2^n}.
\]

Solution: Basis step: \( P(1) : \frac{1}{2} = \frac{2^2 - 2 - 1}{2}, \) which is true.

For the inductive step, assume

\[
\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \ldots + \frac{k}{2^k} = \frac{2^{k+1} - 2 - k}{2^k}
\]

and prove

\[
\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \ldots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} = \frac{2^{k+2} - 2 - (k + 1)}{2^{k+1}}.
\]

Note that

\[
\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \ldots + \frac{n}{2^k} + \frac{k + 1}{2^{k+1}} = \frac{2^{k+1} - 2 - k}{2^k} + \frac{k + 1}{2^{k+1}} = \frac{2^{k+2} - 4 - 2k + (k + 1)}{2^{k+1}} = \frac{2^{k+2} - 2 - (k + 1)}{2^{k+1}}.
\]
Show all necessary work in each problem to receive credit. Please turn in well-organized work and complete solutions. You may only use your 1 page cheat sheet. The take home problem can replace one problem on this exam.

1. (25 points) True or false (no need to justify):

(a) Every strong induction problem needs two or more base cases.
   Solution: False, some of them only require one (see example 2 page 285).

(b) The recurrence \( a_n = 2a_{n-1} - \sqrt{2}a_{n-2} + a_{n-3} \) with \( a_0 = 0, a_1 = 2 \) and \( a_2 = 3 \) is a linear homogeneous recurrence with constant coefficients of degree 3.
   Solution: True

(c) \( a_n = 2^{n+3} - 5 \) is a solution of \( a_n = a_{n-1} + 5 \)
   Solution: False

(d) The recurrence \( a_n = 100a_{n-1} + 1 \) describes the following sequence: 1, 101, 10101, 1010101, \ldots (these are not binary expansions, rather natural numbers)
   Solution: False, since there is no initial step (if the initial step is \( a_0 = 1 \), then it would be true).

(e) Let \( n \geq 0 \) and

\[
P(n) : \sum_{i=n}^{i=2n-1} (2i + 1) = 3n^2.
\]

Is it true that \( P(17) \) is the following?

\[
P(17) : \sum_{i=0}^{i=33} (2i + 1) = 3 \cdot (17)^2.
\]

I am not asking you to verify \( P(17) \), rather to check whether I wrote the correct formula given \( P(n) \) and \( n = 17 \).
Solution: False, since the lower bound on the sum should be 17
2. (25 points) Solve \( a_n = \frac{a_{n-2}}{9} + \left(\frac{1}{3}\right)^n \) for \( n \geq 2 \) with \( a_0 = 0 \) and \( a_1 = 1 \)

Solution: The characteristic equation for the associated homogeneous equation is \( r^n = \frac{r^{n-2}}{9} \) or \( r^2 = 1/9 \). And so \( r = \pm \frac{1}{3} \). Thus the solution to the homogeneous recurrence is

\[
a_n = \alpha \left(\frac{1}{3}\right)^n + \beta \left(-\frac{1}{3}\right)^n.
\]

A particular solution is \( \gamma n \left(\frac{1}{3}\right)^n \), so plugging it in \( a_n = \frac{a_{n-2}}{9} + \left(\frac{1}{3}\right)^n \) we have

\[
\gamma n \left(\frac{1}{3}\right)^n = \frac{1}{9} (\gamma (n-2) \left(\frac{1}{3}\right)^{n-2}) + \left(\frac{1}{3}\right)^n
\]

\[
\gamma n \left(\frac{1}{3}\right)^n = \gamma (n-2) \left(\frac{1}{3}\right)^2 \left(\frac{1}{3}\right)^{n-2} + \left(\frac{1}{3}\right)^n
\]

\[
\gamma n = \gamma (n-2) + 1.
\]

Thus \( \gamma = \frac{1}{2} \). So a particular solution of the nonhomogeneous recurrence is \( \frac{1}{2} n \left(\frac{1}{3}\right)^n \).

A solution of the nonhomogeneous recurrence is

\[
a_n = \alpha \left(\frac{1}{3}\right)^n + \beta \left(-\frac{1}{3}\right)^n + \frac{1}{2} n \left(\frac{1}{3}\right)^n
\]

Now \( n = 0 \rightarrow 0 = \alpha + \beta \)

and \( n = 1 \rightarrow 1 = \alpha (1/3) + \beta (-1/3) + \frac{1}{6} \).

And so \( \alpha = -\beta \), and replacing it in the second equation above implies that \( \alpha = \frac{5}{42} \) and \( \beta = -\frac{5}{4} \). Therefore have that

\[
a_n = \frac{5}{4} \left(\frac{1}{3}\right)^n - \frac{5}{4} \left(-\frac{1}{3}\right)^n + \frac{1}{2} n \left(\frac{1}{3}\right)^n , n \geq 0.
\]
3. (25 points) Use mathematical induction to show that $P(n) : n^3 > n^2 + 3$, for $n \geq 2$.
Solution: Basis step: $P(2) : 2^3 > 2^2 + 3$, which is obviously true.
Inductive step: Assume that $P(k) : k^3 > k^2 + 3$, and prove that $P(k + 1) : (k + 1)^3 > (k + 1)^2 + 3$, for $k \geq 2$. Note that $(k + 1)^3 = (k + 1)(k + 1)^2 = k(k + 1)^2 + (k + 1)^2 \geq 2(k + 1)^2 + (2 + 1)^2 > (k + 1)^2 + 9 > (k + 1)^2 + 3$.

4. (25 points) Some problem out of Sections 7.4 and/or 7.5

(EXTRA CREDIT: 5 points) Find a recurrent definition that counts the number of ways to climb an $n$-step staircase, if you go up either one or three steps at a time (assume that $n$ is a multiple of 3).
Solution: $a_n = a_{n-1} + a_{n-3}$ with $a_1 = 1, a_2 = 1, a_3 = 3$. 
