MA 3025, Sample Exam 1, page 1

Solve 4 of the 5 problems (25 points each) or all five of them (20 points each). You may solve all 5 and decide to turn in 4 (please cross off the one you don’t want me to grade then).

1. [25/20 Pts] In the questions below suppose $A = \{x, y\}$ and $B = \{x, \{x\}\}$. Mark each statement TRUE or FALSE, no explanations are needed.

(a) $x \subseteq B.$
    Ans: False since $x$ is not a set (note that $x \in B$)

(b) $\emptyset \in P(B)$, where $P(B)$ is the power set of $B$.
    Ans: True, because the power set of any set has the element empty set in it.

(c) $\{x\} \subseteq A - B$.
    Ans: False since $A - B = \{y\}$

(d) $|P(A)| = 4$, $P(A)$ is the power set of $A$.
    Ans: True since $|P(A)| = 2^{|A|} = 4$

2. [25/20 Pts] (a) Find $\bigcup_{i=1}^{\infty} (i, \infty)$.
    Ans: $(1, \infty)$

    (b) Find gcd(78, 35)
    
    $78 = 2 \cdot 35 + 8$
    $35 = 4 \cdot 8 + 3$
    $8 = 2 \cdot 3 + 2$
    $3 = 1 \cdot 2 + 1$
    
    Ans: 7

    (c) Find the coefficient of $y^6$ in the expansion of $(2x - y)^{11}$.
    Ans: $\binom{11}{6} 2^5 (-1)^6 x^5 = 14784x^5$ since we would look at $(2x - y)^{11}$ as a function of $y$ only.

    (Note that generally the coefficient of $x^5 y^6$ is $\binom{11}{5} 2^5 (-1)^6$ or $\binom{11}{5} 2^5 (-1)^6 = 14,784$.

    The number “11 choose 6” is the binomial coefficient from the Pascal’s triangle.)

    (d) How many bit strings of length 6 begin with 1 or end with 1
    Ans: $2^5 + 2^5 - 2^4 = 48$
3. [25/20 Pts] Prove or disprove: The average of two rational numbers is rational.

Ans: True.

Proof: Let \( x \) and \( y \) be the two rational numbers. Then \( x = \frac{a}{b} \) and \( y = \frac{c}{d} \) with \( a, b, c, d \) integers and \( b \neq 0, d \neq 0 \). Then the average of \( x \) and \( y \) is \( \frac{x + y}{2} = \frac{\frac{a}{b} + \frac{c}{d}}{2} = \frac{ad + bc}{2bd} \). Since \( ad + bc \) and \( 2bd \) are integers, and also \( 2bd \neq 0 \), we have that the average is rational.

\[ \square \]

4. [25/20 Pts] Find all solutions to the system of congruences:

\[ x \equiv 7 \pmod{10} \]
\[ x \equiv 4 \pmod{11} \]

Solution: We use the Chinese Remainder Thm. with \( m = 110 \).

\[ a_1 = 7, \quad M_1 = 11, \quad y_1 = 1 \]
\[ a_2 = 4, \quad M_2 = 10, \quad y_2 = 10 \]

And so the solution is \( x = 7 \cdot 11 \cdot 1 + 4 \cdot 10 \cdot 10 \pmod{110} = 37 \pmod{110} \), or \( x = 110k + 37 \), where \( k \) is an integer.

5. [25/20 Pts] Use a combinatorial proof to show that

\[ \binom{3n}{3} = \binom{2n}{3} + \binom{n}{3} + n \cdot \binom{2n}{2} + 2n \cdot \binom{n}{2} \]

Proof: The number on the left hand side is the number of 3-subsets of a 3n-set. For the right hand side, let the 3n-set contain say, 2n red and n blue elements. There are \( \binom{2n}{3} \) red 3-subsets, \( \binom{2n}{2} \) blue 3-subsets, \( \binom{n}{3} \) red and 1 blue subsets, and \( \binom{2n}{1} \) red and 2 blue subsets. We thus have a total of \( \binom{2n}{3} + \binom{n}{3} + n \cdot \binom{2n}{2} + 2n \cdot \binom{n}{2} \) subsets.

\[ \square \]