7 Advanced Counting Techniques

7.5 Inclusion-Exclusion

1. The principle of Inclusion-Exclusion: Let \( A_1, A_2, \ldots, A_n \) be finite sets. Then

\[
|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|.
\]

2. if \( n = 2 \):

\[
|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|
\]

and if \( n = 3 \):

\[
|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|
\]

7.6 Applications of Inclusion-Exclusion

1. if we’re counting the number \( N(P'_1, P'_2, \ldots, P'_n) \) of elements that do not have properties \( P_i \) \( (1 \leq i \leq n) \) we can use the following: let \( A_i \) be the subset counting the elements that have the property \( P_i \) \( (1 \leq i \leq n) \), and so the number of elements without any properties \( P_i \) is \( N - |A_1 \cup A_2 \cup \ldots \cup A_n| \), where \( N \) is the total number of elements in the set (the value \( |A_1 \cap A_2 \cap \ldots \cap A_n| \) is denoted by \( N(P_1, P_2, \ldots, P_n) \) and it represents the number of elements that have the properties \( P_i \) \( (1 \leq i \leq n) \))

2. The Sieve of Eratosthenes is used to find all the primes not exceeding a specified positive integer \( n \): list all the natural numbers between 2 and \( n - 1 \) (inclusive), and then keep the first prime number but delete its multiples, then keep the second prime number but delete its multiples, \ldots up to the largest prime number that is less than or equal to \( n \). To use the inclusion-exclusion principle: let \( P_1 \) be the statement that the first prime number divides \( n \), \( P_2 \) be the statement that the second prime number divides \( n \), \ldots, \( P_k \) be the statement that the \( k \)th prime number divides \( n \) (where the last prime number, \( k \), is at most \( \sqrt{n} \)). Then

\[
N(P'_1 P'_2 \ldots P'_k) = (n - 1) - N(P_1) - N(P_2) - \ldots - N(P_k) + N(P_1 P_2) + N(P_1 P_3) + N(P_2 P_3) + \ldots N(P_{k-1} P_k) - N(P_1 P_2 P_3) - \ldots - N(P_{k-2} P_{k-1} P_k) + N(P_1 P_2 P_3 P_4) \ldots
\]

3. The number of onto functions from a set with \( m \) elements to a set with \( n \) elements: if we let \( P_k \) denote the property that the value \( k \) is not in the range (i.e.
there is no value \( x \) of the domain that gets mapped to \( k \), then

\[
N(P'_1 P'_2 \ldots P'_k) = N - N(P_1) - N(P_2) - \ldots - N(P_k) \\
+ N(P_1 P_2) + N(P_1 P_3) + N(P_2 P_3) + \ldots N(P_{k-1} P_k) \\
- N(P_1 P_2 P_3) - \ldots - N(P_{k-2} P_{k-1} P_k) \\
+ N(P_1 P_2 P_3 P_4) \ldots \\
= n^m - C(n, 1)(n - 1)^m + C(n, 2)(n - 2)^m - \ldots + (-1)^{n-1} C(n, n - 1)1^m
\]

4. a derangement is a permutation of \( n \) objects that leaves no objects in their original position (i.e. when permuting the elements, every element needs to change its position). The number of derangements of \( n \) elements is \( D_n \) (let \( P_i \) be the permutation that fixes element \( i \) \((1 \leq i \leq n)\)), and count \( D_n = N(P'_1 P'_2 \ldots P'_n) \) using the inclusion-exclusion principle)

\[
D_n = n! - \binom{n}{1} (n - 1)! + \binom{n}{2} (n - 2)! - \binom{n}{3} (n - 3)! + \ldots + (-1)^n \binom{n}{n} (n - n)!
\]

\[
D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right)
\]