Show all necessary work in each problem to receive credit.

1. (20 points) Create a truth table for $(p \land (p \rightarrow q)) \rightarrow q$. Is the statement a tautology or contradiction or neither?

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P \rightarrow Q</th>
<th>P \land (P \rightarrow Q)</th>
<th>(P \land (P \rightarrow Q)) \rightarrow Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
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<td>T</td>
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<td>F</td>
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<td>T</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Proof: Yes, it is a tautology.

Method 2

\[
\begin{array}{c}
\frac{P \land (P \rightarrow \bar{q})}{T} \\
\frac{P \land (P \rightarrow \bar{q})}{F}
\end{array}
\xrightarrow{F} \frac{q}{\bar{q}}
\]

Not possible! since $P \rightarrow \bar{q}$ is $\bar{F}$, it follows that $P \land (P \rightarrow \bar{q})$ is $\bar{F}$.

2. (20 points) Show that $p \rightarrow (q \rightarrow r)$ is logically equivalent to $\neg(p \rightarrow \neg q) \rightarrow r$ without using truth tables.

\[
\neg(p \rightarrow \neg q) \rightarrow r \\equiv (p \rightarrow \neg q) \lor r \\
\equiv (\neg p \lor \neg q) \lor r \\
\equiv \neg p \lor (\neg q \lor r) \\
\equiv p \lor (q \rightarrow r) \\
\equiv p \rightarrow (q \rightarrow r).
\]
3. (20 points) Do both (1) and (2) below (Only the first two sentences in each part will be graded if more than two are turned in).

1. Translate 2 of the following 3 sentences.

(a) Let $p$: You get a B in the final, $q$: You solve every homework problem assigned, and $r$: You pass the class.
Let $S$ be the set of all students currently enrolled in MA 2025.
Write out the proposition using $p, q, r$ and logical connectives:
Every student currently enrolled in the MA 2025 class that solves every homework problem and will get a B in the final, will pass the class.

$$\forall x \in S \left( (q(x) \land p(x)) \rightarrow r(x) \right)$$

(b) Let $E(x, y)$ be the predicate that $x$ sent an email to $y$. Use quantifiers to express:
Somebody sent an email to exactly one person.

$$\exists x \exists! y \ E(x, y)$$

(c) Let $E(x, y)$ be the predicate that $x$ sent an email to $y$. Use quantifiers to express:
There is at least one person that didn’t send any emails.

$$\exists x \forall y \neg E(x, y)$$

2. Negate the following 2, and write your answer in a form such that the negation directly precedes the predicates:

(a) $\exists x \forall y \left( x^2 + y^2 > 1 \right)$

$$\forall x \exists y \left( x^2 + y^2 \leq 1 \right)$$

(b) $\exists x \exists y \left( \left( \neg p(x, y) \rightarrow q(x, y) \right) \land r(x, y) \right)$

$$\exists x \exists y \left( \left( p(x, y) \lor q(x, y) \right) \land \neg r(x, y) \right)$$
4. (20 points) For $y \neq 0$, let $Q(x, y) : \frac{x}{2y} = 1$.

Determine the truth value of each of these statements (explain why it is true or false):

1. $\forall x \exists y Q(x, y)$ (F) Let $y = \frac{x}{2}$. Then $\frac{x}{2y} = \frac{x}{\frac{x}{2}} = \frac{x}{x} = 1$, $\forall x$.
   Note that if $x = 0$, then $\frac{0}{2y} \neq 1$.
   This part will get you partial credit since it would be easy to miss $x = 0$.

2. $\exists y \forall x Q(x, y)$ (F) No $y$ value works for every $x$ since you must choose $y$ first.

3. $\exists x Q(x, 1)$ (T) Let $x = 2$. Then $Q(2, 1) : \frac{2}{2 \cdot 1} = 1$ true

4. $\exists y Q(0, y)$ (F) Since $Q(0, y) : \frac{0}{2y} = 1$ which is never true as 0 divided by any nonzero number is always zero.
5. (20 points) Determine if the following two arguments are correct or not (explain what inference rules were used if correct, or point out where the argument fails if incorrect):

1. Let $p$ and $q$ be statements.
   (a) $\exists x \ p(x) \land \exists x \ q(x)$ \hspace{1cm} \text{premise}
   (b) $p(c)$ \hspace{4cm} \text{true using simplification & existential instantiation}
   (c) $q(d)$ \hspace{4cm} \text{true using addition rule}
   (d) $p(c) \lor q(d)$ \hspace{4cm} \text{true using existential generalization}
   (e) $\exists x (p(x) \lor q(x))$ \hspace{4cm} \text{true using addition rule}

   \underline{Argument is true}

2. Premise 1: If an integer $x \geq 2$, then $x^3 \geq 8$.

   Premise 2: Let $x < 2$.

   Conclusion: Therefore $x^3 < 8$.

   \underline{Argument is false. Fallacy of denying hypothesis.}