5 Induction and Recursion

5.1 Mathematical induction (weak induction)

This section presents another proof technique (besides direct, contrapositive and contradiction)

1. mathematical induction is used to prove statements that are true for all positive integers or for positive integers greater than some number \( n_0 \).

2. the proof of a statement \( P(n), n \geq 1 \) has two steps:

   (a) **Basis Step**: show that \( P(1) \) is true. The basis step is the foundation of the proof. For example, a statement can ask to prove that: for all positive integers we have that \( n! \leq n^n \), and so the proof will start with:
   
   **Proof**: Show \( P(1) \) is true: Note that \( 1! = 1 \) is less than or equal to \( 1^1 = 1 \). The inductive step will continue now.

   (b) **Inductive Step**: \( \forall k \geq 1 \ (P(k) \rightarrow P(k + 1)) \) This step proves that every statement \( P(k + 1) \) is true given that the previous one, \( P(k) \), is true. This implication is proved using a direct proof (\( P(k) \) the inductive hypothesis).

3. So here is what the induction does: if \( P(1) \) is T as we showed in the basis step, we can use the inductive step we obtain that \( P(2) \) is T (here \( k = 1 \)). And then, since we have that \( P(2) \) is T, we use the inductive step to get that \( P(3) \) is T. And since \( P(3) \) is T, we again use the inductive step to get that \( P(4) \) is T. And continuing in this manner, one may see how \( P(n) \) is T by building up on the basis step and proving that we can go from any \( P(k) \) to \( P(k + 1) \).

4. The basis step may not always be exactly \( P(1) \). For example, if the results says to show that a statement is true for the positive integers greater than or equal to some number \( n_0 \), then \( P(n_0) \) is what we have to show in the basis step (\( P(1) \) is common, but it could be some \( P(n_0) \) for \( n_0 > 1 \)). For example, if the result states that: \( \forall n > 2 \) prove that \( n! \leq n^n \), then the basis step will check \( P(3) \):

   Note that \( 3! = 3 \cdot 2 \cdot 1 = 6 \) is less than or equal to \( 3^3 = 27 \).

5. types of problems that we will prove by mathematical induction:

   - summations (day 1)
   - inequalities (day 2)
   - divisibility (day 3)
   - sets