5 Counting

5.3 Permutations and Combinations

1. recall: for integers \( n \geq 0 \), the factorial \( f(n) = n! \) is defined by

\[
n! = \begin{cases} 
1, & \text{if } n = 0; \\
(n - 1)!n, & \text{if } n > 0.
\end{cases}
\]

2. a permutation is an ordering, or arrangement, of the elements in a finite set:

Definition: A permutation \( \pi \) of \( A = \{a_1, a_2, \ldots, a_n\} \) is an ordering \( a_{\pi_1}, a_{\pi_2}, \ldots, a_{\pi_n} \) of the elements of \( A \) (no repeats in the list). Example: a permutation of \( A = \{1, 2, 3\} \) is 1, 3, 2

3. there are \( n! \) permutations of an \( n \)-element set (an \( n \)-element set is also called an \( n \)-set).

4. an \( r \)-permutation of an \( n \)-set \( A \) \((r \leq n)\) is an ordering \( a_{\pi_1}, a_{\pi_2}, \ldots, a_{\pi_r} \) of some \( r \)-subset of \( A \). Example: a 3-permutation of the 4-set \( A = \{1, 2, 3, 4\} \) is 2, 4, 3, and a different one is 2, 3, 4 (since they are sequences and so the order matters).

5. there are \( P(n, r) \) of these:

\[
P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1) = \frac{n(n - 1) \cdots (n - r + 1) \cdots (2)(1)}{(n - r)(n - r - 1) \cdots (2)(1)} = \frac{n!}{(n - r)!}.
\]

6. Example: The number of 3-digit decimal numbers with no repeated digit is \( P(10, 3) = 720 \) (leading zeros allowed). This could also be done using the product rule: \( 10 \cdot 9 \cdot 8 = 720 \)

7. Example: The number of 3-digit decimal numbers with repetition (and leading zeros) allowed is by the product rule \( 10^3 = 1000 \)

8. an \( r \)-combination of an \( n \)-set \( A \) \((r \leq n)\) is an \( r \)-subset \( \{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\} \) of the \( n \)-set \( A \). Example: a 3-combination of the 4-set \( A = \{1, 2, 3, 4\} \) is \( \{2, 4, 3\} \) (which is the same as the 3-combination \( \{2, 3, 4\} \) since \( \{2, 4, 3\} = \{2, 3, 4\} \), as the order in a set does not matter).

9. there are \( C(n, r) \) of these. The number \( C(n, r) \) is also commonly written \( \binom{n}{r} \), which is called a binomial coefficient. These are associated with a mnemonic called Pascal’s Triangle and a powerful result called the Binomial Theorem, which makes it simple to compute powers of binomials. (The inductive proof that the binomial theorem is a bit messy, and it becomes easier if it uses the idea of combinatorial proof—see MA 3025. A combinatorial proof that we work with here consists of arguing that both sides of an equation of two integer expressions are equal to the cardinality of the same set.)
10. note that we could construct an $r$-permutation of an $n$-set in two steps: first take an $r$-combination, then take a permutation of the $r$-combination. It follows by the Product Rule that $P(n, r) = r!C(n, r)$, but then

$$C(n, r) = \frac{1}{r!} P(n, r) = \frac{n!}{r!(n-r)!}.$$ 

This is not a practical formula for hand computation, but we can find a better one without too much difficulty. It looks like this:

$$C(n, r) = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)(n-r)!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!}; \quad (1)$$

note that there are exactly $r$ factors in numerator and denominator alike.

11. $C(n, r) = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!(n-(n-r))!} = C(n, n-r). \quad (2)$

This makes some potentially nasty computations pretty easy to carry out. For example, what is $C(100, 98)$? By definition, $C(100, 98) = \frac{100!}{98!2!}$, which is beyond the range of many calculators. And if we use formula (1) for hand computation of $r$-combinations, we’ll have 98 factors in both numerator and denominator. But by (2) and (1) together, we have

$$C(100, 98) = C(100, 2) = \frac{(100)(99)}{2!} = 50 \cdot 99 = 4,950.$$