5 Counting

5.2 The Pigeonhole Principle

1. The Pigeonhole Principle says that if \( k + 1 \) pigeons fly into \( k \) pigeonholes (or into at most \( k \) pigeonholes), at least one pigeonhole must contain at least two pigeons.

**Theorem 1** (The Pigeonhole Principle) If \( k \) pigeons occupy \( j < k \) pigeonholes, then at least one pigeonhole contains at least two pigeons.

**Example:**

(a) The “hello, world” problem for the pigeonhole principle is the “sock problem”: In your dresser drawer you have a jumble of socks in two colors, say blue and gray. It’s dark, and you don’t want to wake your spouse. How many socks must you grab to guarantee that you have a pair of the same color?

**Solution:** Three socks suffice. You might end up with three blue, or three gray, but with only two colors you’re guaranteed to have at least two blue or at least two gray.

(b) Show that in a group of eight people there must be two whose birthdays fall on the same day of the week.

**Solution:** The pigeons are the people in the group, and the pigeonholes are the days of the week. Since there are 8 people and 7 days, two people must share a day.

The first generalization of the principle is this:

**Theorem 2** If \( n \) pigeons occupy \( k \) pigeonholes, then at least one pigeonhole contains at least \( \lceil n/k \rceil \) pigeons.

We can use this version to answer more difficult questions: What is the smallest \( n \) such that at least one of \( k \) boxes must contain at least \( r \) of \( n \) objects? By Theorem 2, in order to have at least \( r \) objects into a box, we need

\[
\lceil n/k \rceil \geq r \\
n/k > r - 1 \\
n > k(r - 1).
\]

So the smallest integer \( n \) that forces some box to contain \( r \) of \( n \) objects is \( n = k(r - 1) + 1 \).