Appendix A1: Axioms for the Real Numbers and the Positive Integers

1. be able to use the laws for the reals: closure (stay within the same set), associative and commutative, additive and multiplicative, identity (the zero is the identity for addition, and one is the identity for multiplication), and the inverses (additive is $-x$ and the multiplicative one is $\frac{1}{x}$), distributive law

2. order axioms:
   - for any two reals $x$ and $y$, exactly one of the following must be true:
     (a) $x > y$
     (b) $x = y$
     (c) $x < y$
   - $\forall x, y, z \in \mathbb{R} (x > y$ and $y > z$, then $x > z)$. Note that $x > y$ and $z > y$ does not imply that $x > z$
   - $\forall x, y, z \in \mathbb{R} (x > y$, then $x + z > y + z)$.
   - $\forall x, y, z \in \mathbb{R} (x > y$ and $z > 0$, then $xz > yz)$. Note that if $z < 0$ then $xz < yz$, and if $z = 0$ then $xz = 0 = yz$

3. a number $a$ is an upper bound for a set $S$, if $\forall x \in S (x \leq a)$

4. a number $b$ is a lower bound for a set $S$, if $\forall x \in S (x \geq b)$

5. Completeness Property of the reals: Every nonempty set has a lower and an upper bound. For example, for the set $S = \{0, 3, -5, 12/7\}$, a lower bound is $a = -5$ (note that we mentioned “a lower bound”, since any number smaller that $-5$ will also be a lower bound)

Proving basic known facts using the above axioms:

1. the additive/multiplicative element is unique (note that you may use this result after you prove it to be true, and the same is true for all statement below)

2. the additive/multiplicative inverse is unique

3. $\forall x \in \mathbb{R} (x \cdot 0 = 0 = 0 \cdot x)$

4. $\forall x \in \mathbb{R} (x \cdot y = 0 \rightarrow (x = 0 \lor y = 0))$

5. $1 > 0$

6. $\forall r \in \mathbb{R}, \exists n \in \mathbb{Z} (n > r)$ (that means you can always find an integer that is larger that the particular chosen real number)

7. WOP (Well Ordering Principle): Every nonempty subset of the real numbers has a smallest element.
8. Math Induction Axiom: If $S$ is a set with

- $1 \in S$, and
- $(n \in S) \rightarrow ((n + 1) \in S)$

then $S = \mathbb{Z}$