1. (15 points) For the two matrices \( A = \begin{bmatrix} -2 & 3 & 2 \\ 2 & -2 & -1 \\ -4 & 5 & 3 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 3 & 2 \\ -1 & -2 & -1 \\ 2 & 5 & 3 \end{bmatrix} \).

\( (a) \) find \( A^T = \begin{bmatrix} -2 & 2 & -4 \\ 3 & -2 & 5 \\ 2 & -1 & 3 \end{bmatrix} \)

\( (b) \) find \( A - B = \begin{bmatrix} -3 & 0 & 0 \\ 3 & 0 & 0 \\ -6 & 0 & 0 \end{bmatrix} \).

\( (c) \) find \( AB = \begin{bmatrix} -1 & -2 & -1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{bmatrix} \).
2. (15 points) Find the Row Echelon Form of the coefficient matrix of the following system

\[
\begin{align*}
  x_1 + x_2 - x_3 &= 0 \\
-2x_1 - 2x_2 + 5x_3 &= 1 \\
8x_1 + 2x_2 - 13x_3 &= -3
\end{align*}
\]

Solution:

\[
\begin{bmatrix}
  1 & 1 & -1 \\
-2 & -2 & 5 \\
 8 & 2 & -13
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  1 & 1 & -1 \\
 0 & 0 & 3 \\
 0 & -6 & -5
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  1 & 1 & -1 \\
 0 & 1 & \frac{5}{6} \\
 0 & 0 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  1 & 1 & -1 \\
 0 & 1 & \frac{5}{6} \\
 0 & 0 & 1
\end{bmatrix}
\]
3. (20 points) **Choose one of the two systems below.** Use Gaussian (or Gauss-Jordan elimination if you want) to find the solution of the system. Indicate whether the system is: inconsistent, consistent, underdetermined, overdetermined, determined, homogeneous.

(a)

\[-x_1 - 5x_2 = 2\]
\[2x_1 - 3x_2 = 1\]
\[-5x_1 + x_2 = -3\]

\[
\begin{bmatrix}
-1 & -5 & 2 \\
2 & -3 & 1 \\
-5 & 1 & -3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 5 & -2 \\
0 & -13 & 5 \\
0 & 26 & -13
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 5 & -2 \\
0 & 1 & \frac{5}{13} \\
0 & 0 & -3
\end{bmatrix}
\]

Inconsistent, overdetermined system.

(b)

\[x_1 - x_2 + x_3 = 0\]
\[2x_1 - 4x_2 + x_3 = 0\]
\[ -4x_1 - 2x_2 = 0\]

\[
\begin{bmatrix}
1 & -1 & 1 & 0 \\
2 & -4 & 1 & 0 \\
-4 & -2 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & 1 & 0 \\
0 & -2 & -1 & 0 \\
0 & -6 & 4 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & 1 & 0 \\
0 & 1 & \frac{1}{2} & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & 1 & 0 \\
0 & 1 & \frac{1}{2} & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

System is consistent, determined, and homogeneous with solution \(x_1 = x_2 = x_3 = 0\).
4. (15 points) Let \( A \) and \( B \) be the two matrices below

\[
A = \begin{bmatrix}
-2 & 3 & 2 \\
-4 & 5 & 2
\end{bmatrix},
B = \begin{bmatrix}
1 & 3 & 2 \\
-1 & -2 & -1 \\
2 & 5 & 3
\end{bmatrix}, \text{ and } I_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Without performing the operations, decide whether the following operations can be performed or not. If they can, provide the size of the expected resulting matrix (the size of \( A \) is \( 2 \times 3 \) and \( B \) is \( 3 \times 3 \)):

1. \( A - B \)  
   not possible

2. \( -B \)  
   yes, and the matrix is a 3 by 3 matrix

3. \( A \cdot I_3 \)  
   yes, and the matrix is a 2 by 3 matrix

4. \( B \cdot A \)  
   not possible

5. \( A - I_3 \)  
   not possible

5. (15 points) Consider a linear system whose augmented matrix can be reduced to

\[
\begin{bmatrix}
1 & -\frac{3}{2} & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \alpha + 2 & 0
\end{bmatrix}
\]

For what values of \( \alpha \) will the system have infinitely many solutions?

Soln: If \( \alpha = -2 \) then we get the solution \((0, 0, a)\) where \( a \in R \), which is a set of infinitely many solutions.
6. For the two matrices below $A = \begin{bmatrix} -2 & 3 & 2 \\ 2 & -2 & 1 \\ -4 & 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 2 \\ -1 & -2 & -1 \\ 2 & 5 & 3 \end{bmatrix}$.

(a) (20 points) Find $A^{-1}$ using the elementary row operations (show your work, not just the final answer)

$$A^{-1} = \begin{bmatrix} \frac{-2}{3} & \frac{3}{2} & \frac{2}{3} \\ 2 & -2 & 1 \\ -4 & 5 & 2 \end{bmatrix} \xrightarrow{R_2 + R_1; R_3 - 2R_1} \begin{bmatrix} \frac{-2}{3} & 3 & \frac{2}{3} \\ 0 & 1 & \frac{3}{2} \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{R_2; R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ \frac{-3}{3} & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 - 3R_3; R_1 + R_3} \left[ \begin{array}{ccc} 1 & \frac{-2}{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \xrightarrow{R_1 + \frac{3}{2}R_2} \left[ \begin{array}{ccc} 1 & 0 & 0 \\ \frac{9}{2} & -2 & \frac{-7}{2} \\ 0 & 1 & 0 \end{array} \right] \xrightarrow{R_3 - \frac{7}{2}R_2} \left[ \begin{array}{ccc} 1 & 0 & 0 \\ \frac{9}{2} & -2 & \frac{-7}{2} \\ 0 & 1 & 0 \end{array} \right]$$

Thus $A^{-1} = \begin{bmatrix} \frac{9}{2} & -2 & \frac{-7}{2} \\ 4 & -2 & -3 \\ -1 & 1 & 1 \end{bmatrix}$.
CREDIT(5 points) solve $A \cdot X = B$ by using part (a) above

$$X = A^{-1}B = \begin{bmatrix} \frac{9}{2} & -2 & \frac{-7}{2} \\ 4 & -2 & -3 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 2 \\ -1 & -2 & -1 \\ 2 & 5 & 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$