MA1114  Exam # 2

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Instructor: Dr. Ralucca Gera

Name ____________________________  Key

time your class meets _____________

Show all necessary work in each problem to receive credit. You may not obtain more than 100 points on the exam.

1. (20 points) Determine \( \int \frac{1}{x(x-2)^2} \, dx \).

\[
\frac{1}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}
\]

\[
= A(x-2)^2 + Bx(x-2) + C(x)
\]

\[
|X=2: \quad 1 = 0 \quad \bigoplus \quad \bigoplus \quad C = \frac{1}{3}
\]

\[
|X=0: \quad 1 = 4A \quad \bigoplus \quad \bigoplus \quad 4 = \frac{1}{4}
\]

\[
|X=1: \quad 1 = \frac{1}{4} - B + \frac{1}{2} \rightarrow \quad B = -\frac{1}{4}
\]

\[
= \frac{1}{4} \int \frac{1}{x} \, dx - \frac{1}{4} \int \frac{1}{x-2} \, dx + \frac{1}{2} \int \frac{1}{x-2} \, dx = \frac{1}{4} \ln |x| - \frac{1}{4} \ln |x-2| - \frac{1}{2} \ln |x-2| + C
\]

2. (20 points) Determine whether \( \int_1^\infty \frac{1}{(2x-1)^2} \, dx \) converges or diverges. If it converges give its value.

\[
\int_1^\infty \frac{1}{(2x-1)^2} \, dx = \lim_{t \to \infty} \int_1^t \frac{1}{(2x-1)^2} \, dx
\]

\[
= \lim_{t \to \infty} \left[ -\frac{1}{2x-1} \right]_1^t
\]

\[
= \lim_{t \to \infty} \left[ -\left( \frac{1}{2t-1} - \frac{1}{2 \times 1 - 1} \right) \right]
\]

\[
= \frac{1}{2} \quad \text{convergent}
\]
Solve either 4 problems (15 points each) or all 6 (10 points each) problems below. Please identify what test you have used.

3. (15 points/ 10 points) Determine whether the series converges or diverges: $\sum_{n=1}^{\infty} \frac{n^3}{4^n}$

$$\lim_{n \to \infty} \left| \frac{(n+1)^3}{4^{n+1}} \cdot \frac{n^3}{4^n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^3}{n^3} \cdot \frac{1}{4} \right| = \frac{1}{4} < 1 \text{ thus series converges by the ratio test (it actually converges absolutely). Divergence test together with L'Hopital works as well.}$$

4. (15 points/ 10 points) Determine whether the series converges or diverges: $\sum_{n=1}^{\infty} \frac{n}{\ln(n+1)}$

$$\lim_{n \to \infty} \frac{n}{\ln(n+1)} \overset{L'Hopital}{=} \lim_{n \to \infty} \frac{1}{\frac{1}{n+1}} = \infty \neq 0$$

thus series diverges by the divergence test
5. (15 points/ 10 points) Find a formula for the following series \(-9 + 3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} \ldots\)

and determine whether it converges or diverges:

\[-9 + 3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} \ldots = -9 \left(1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} - \frac{1}{3^5} \ldots\right)\]

\[= \sum_{n=1}^{\infty} -9 \cdot \left(\frac{1}{3}\right)^{n-1}\] which is a geometric series

with \(|r| = \frac{1}{3} < 1\), thus convergent.

Alternating series test works as well.

6. (15 points/ 10 points) Determine whether the sequence converges or diverges:

\[a_n = e^{-n} \cos^2 n = \frac{\cos^2 n}{e^n}\]

Note that

\[0 \leq \cos^2 n \leq 1\]

so we have

\[0 \leq \frac{\cos^2 n}{e^n} \leq \frac{1}{e^n}\]

\[\lim_{n \to \infty} \frac{\cos^2 n}{e^n} = 0\]

thus by Squeeze Theorem, and so

the sequence converges

\[\lim_{n \to \infty} e^n = 0\]
7. (15 points/ 10 points) Determine whether the series converges or diverges: \[ \sum_{n=1}^{\infty} \frac{2n^{17}}{7n^{18} + 2} \]

\[ \lim_{n \to \infty} \frac{\frac{2n^{17}}{7n^{18} + 2}}{\frac{1}{n}} = \frac{2}{7} \neq 0 \] and so by the limit comparison test, the original series diverges since the harmonic series diverges. Integral test works as well.

8. (15 points/ 10 points) Determine whether the series converges or diverges: \[ \sum_{n=1}^{\infty} \frac{2 + \sin n}{n^2} \]

\[ 0 \leq \frac{2 + \sin n}{n^2} \leq \frac{3}{n^2} \text{ So by comparison theorem the series converges as the } p - \text{series } \sum_{n=1}^{\infty} \frac{3}{n^2} \text{ converges.} \]

(Note that Squeeze theorem gives the limit zero, but it does not help with the series.)
Extra credit (5 points) Determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{(2n)!}{4^n}$$

$$\lim_{n \to \infty} \left| \frac{(2n+2)!}{4^{n+1}} \right| = \lim_{n \to \infty} \left| \frac{(2n+2)(2n+1)}{4} \right| = \infty$$

Thus series diverges by the root test.