1 Chapter 1: Matrices and Systems of Equations

1.5 Elementary Matrices

1. Elementary matrices, denoted by $E$, are matrices obtained by performing one elementary row operation on $I$. There are three types:

- **[Type I]** interchanging two rows:
  \[
  \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & 1 \\
  0 & 1 & 0
  \end{pmatrix}
  \]

- **[Type II]** constant multiple of a row:
  \[
  \begin{pmatrix}
  1 & 0 & 0 \\
  0 & -3 & 0 \\
  0 & 0 & 1
  \end{pmatrix}
  \]

- **[Type III]** adding a multiple of a row to another row:
  \[
  \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 2 & 1
  \end{pmatrix}
  \]

2. post multiplying a matrix $B$ by an elementary matrix $E$ is equivalent to performing that particular column operation on $B$

3. pre multiplying a matrix $B$ by an elementary matrix $E$ is equivalent to performing that particular row operation on $B$

4. that is, every elementary matrix has an inverse which is also an elementary matrix:
   - (a) the inverse of a matrix of type I is also of type I, actually it is its own inverse
     \[
     \begin{pmatrix}
     1 & 0 & 0 \\
     0 & 0 & 1 \\
     0 & 1 & 0
     \end{pmatrix}
     \]
   - (b) the inverse of a matrix of type II is also of type II:
     \[
     \begin{pmatrix}
     1 & 0 & 0 \\
     0 & -1/3 & 0 \\
     0 & 0 & 1
     \end{pmatrix}
     \]
   - (c) the inverse of a matrix of type III is also of type III:
     \[
     \begin{pmatrix}
     1 & 0 & 0 \\
     0 & 1 & 0 \\
     0 & -2 & 1
     \end{pmatrix}
     \]
5. $B$ is row equivalent to $A$ if multiplying $A$ by a series of elementary matrices we get $B$ (note that every multiplication by an elementary matrix can be viewed as a step in Gaussian Elimination method)

6. The following are equivalent:
   
   (a) $A$ is nonsingular
   (b) $Ax = 0$ has only one solution: $x = 0$
   (c) $A$ is row equivalent to $I$.

7. $Ax = 0$ has a unique solution (namely $0$) $\iff A$ is nonsingular

8. finding $A^{-1}$ using the elementary row operations (page 66).

9. diagonal matrix: if the entries of the diagonal are the only possible nonzero entries of the matrix

10. upper triangular matrix: if the entries below the diagonal are zero (some of the other entries could be zero as well)

11. lower triangular matrix: if the entries above the diagonal are zero (some of the other entries could be zero as well)

12. triangular matrix: if it is either upper or lower triangular matrix (recall strict triangular matrix).

13. skip Triangular Factorization.