1. (10 points) Determine whether \( \int_{0}^{2} \frac{1}{(x-1)^2} \, dx \) converges or diverges. If it converges give its value if it can be found.

Solution: \( \int_{0}^{2} \frac{1}{(x-1)^2} \, dx = \lim_{t \to 1^-} \int_{0}^{t} \frac{1}{(x-1)^2} \, dx + \lim_{s \to 1^+} \int_{s}^{2} \frac{1}{(x-1)^2} \, dx. \)

Since \( \lim_{t \to 1^-} \int_{0}^{t} \frac{1}{(x-1)^2} \, dx = \lim_{t \to 1^-} \left( -\frac{1}{x-1} \right)_{0}^{t} = 1 - \infty = -\infty, \) it follows that the integral \( \lim_{t \to 1^-} \int_{0}^{t} \frac{1}{(x-1)^2} \, dx \) diverges, and so the original integral diverges.

2. (10 points) Determine whether the series \( \sum_{n=1}^{\infty} \frac{n^2 + 1}{4^n} \) absolutely converges, conditionally converges or diverges:

Solution: \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)^2 + 1}{n^2 + 1} \frac{4^n}{4^{n+1}} = \frac{1}{4} < 1, \) thus the series absolutely converges.
3. (10 points) Find \( \int \frac{x^2 + 3x + 1}{x^3 + x} \, dx \)

Solution:

\[
\frac{x^2 + 3x + 1}{x^3 + x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}
\]

\[
x^2 + 3x + 1 = A(x^2 + 1) + (Bx + C)(x)
\]

\[x = 0 \Rightarrow A = 1\]
\[x = 1 \Rightarrow 5 = 2A + B + C \Rightarrow B + C = 3\]
\[x = -1 \Rightarrow -1 = 2A + B - C \Rightarrow B - C = -3\]

Thus \( A = 1, B = 0, \) and \( C = 3. \) So

\[
\int \frac{x^2 + 3x + 1}{x^3 + x} \, dx = \int \frac{1}{x} \, dx + \int \frac{3}{x^2 + 1} \, dx = \ln |x| + 3 \arctan x + C
\]
4. Determine whether the following statements are true or false. If you think the statement is false, give a counterexample (that is, an example that shows the statement is false).

(5 pts) If the sequence \( \{a_n\}_{n=1}^{\infty} \) converges, then \( \lim_{n \to \infty} a_n = 0 \).

False. Counterexample: The series \( \{1^n\}_{n=1}^{\infty} \) converges to 1.

Or the series \( \left\{ \frac{2n}{n + 1} \right\}_{n=1}^{\infty} \) converges to 2.

(5 pts) If \( \lim_{n \to \infty} a_n = 0 \), then \( \sum_{n=1}^{\infty} a_n \) converges.

False. Counterexample: Harmonic series: \( \sum_{n=1}^{\infty} \frac{1}{n} \) since it diverges but \( \lim_{n \to \infty} \frac{1}{n} = 0 \).
(5 pts) If a series is absolutely convergent, then it is convergent.

True

(5 pts) If a series is convergent, then it is absolutely convergent.

False. Counterexample: Alternating harmonic series: \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) since it converges by the Alternating Series Test, but it does not absolute converge.
5. (50 points) Solve 5 of the following 8 problems. Determine whether each sequence converges or diverges. If it converges give its value if it can be found.

(10 pts) \[ \left\{ \frac{n}{\ln(n+1)} \right\}_{n=1}^{\infty} \]

Solution: \[ \lim_{n \to \infty} \frac{n}{\ln(2n+5)} = \lim_{n \to \infty} \frac{1}{\frac{2}{2n+5}} = \infty. \] Thus the sequence diverges by the Divergence Test.

(10 pts) \[ \left\{ (-1)^n n \right\}_{n=1}^{\infty} \]

Solution: \[ \lim_{n \to \infty} \left| \left\{ (-1)^n n \right\}_{n=1}^{\infty} \right| = \lim_{n \to \infty} \frac{n}{3n^2 + e} = 0. \] Thus the sequence converges to 0 by the Alternating Series Test.
(10 pts) \[ \sum_{n=1}^{\infty} \frac{\pi^{n-1}}{(4)^n} \]

Solution: \[ \sum_{n=1}^{\infty} \frac{\pi^{n-1}}{(4)^n} = \sum_{n=1}^{\infty} \frac{1}{4} \cdot \left(\frac{\pi}{4}\right)^{n-1} = \frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^{n-1} \]

which converges since the series \[ \sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^{n-1} \]

converges as a geometric series with \( r = \frac{\pi}{4} \).

In particular, it converges to \( \frac{1}{4} \cdot \frac{1}{1 - \frac{\pi}{4}} = \frac{1}{4 - \pi} \).

(10 pts) \[ \sum_{n=1}^{\infty} \frac{-2}{n^{0.5}} \]

Solution: \[ \sum_{n=1}^{\infty} \frac{-2}{n^{0.5}} = -2 \cdot \sum_{n=1}^{\infty} \frac{1}{n^{0.5}} \]

which is a \( p \)-series with \( p = 0.5 < 1 \) so it diverges.
\[(10 \text{ pts}) \sum_{n=2}^{\infty} \frac{6n + \pi}{\ln n}\]

Solution: \(\lim_{n \to \infty} \frac{6n + \pi}{\ln n} = \lim_{n \to \infty} \frac{6}{\frac{1}{n}} = \infty\) so the series diverges by the divergence test.

\[(10 \text{ pts}) \sum_{n=1}^{\infty} \frac{(-1)^n}{3 + n}\]

Solution: \(b_n = \frac{1}{3 + n} > b_{n+1} = \frac{1}{3 + (n + 1)}\) so the series is decreasing. Also, \(\lim_{n \to \infty} \frac{1}{3 + n} = 0\), so the series converges by the alternating series test.
(10 pts) $\sum_{n=1}^{\infty} \frac{21n^2}{7n^3 + 2}$

Solution: Note that the terms are all positive and so: $\lim_{n \to \infty} \frac{21n^2}{7n^3 + 2} = \frac{21}{7} = 3$. Since $0 < 3 < \infty$, it follows that $\sum_{n=1}^{\infty} \frac{21n^2}{7n^3 + 2}$ diverges, since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

METHOD 2: Solution: Note that the terms are all positive, decreasing and the function $f(n) = \frac{21n^2}{7n^3 + 2}$ is continuous on $[1, \infty)$. Thus $\int_{1}^{\infty} \frac{21n^2}{7n^3 + 2} \, dx = \lim_{t \to \infty} \int_{1}^{t} \frac{21n^2}{7n^3 + 2} \, dx = \lim_{t \to \infty} \ln(7n^3 + 2)|_{1}^{t} = \infty$. So $\sum_{n=1}^{\infty} \frac{21n^2}{7n^3 + 2}$ diverges by the integral test.

(10 pts) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

Solution: Note that the terms are all positive and so: $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges as a $p$ series, and so $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges by the comparison theorem.