Chapter 1: Matrices and Systems of Equations

1.1 Systems of Linear Equations

1. systems of linear equations and their geometric interpretation as the intersection of lines
2. inconsistent system: system has no solutions
3. consistent system: at least one solution
4. equivalent systems: if they have the same set of solutions
5. diagonal entries: the numbers on the diagonal of a matrix $a_{11}, a_{22}, \ldots, a_{n,n}$
6. strict triangular form of a system: the diagonal entries are nonzero, and the entries below the diagonal are all zero
7. coefficient matrix (or simply the matrix) of a system = the coefficients of the variables in system of equations
8. augmented matrix = the coefficient matrix with the additional column of the entries on the right hand side of the system of equations.
9. elementary row operations = matrix operations that are used to solve the original system of equations. They are: Interchanging two rows, multiplying a row by a nonzero constant, and replacing a row by its sum with a multiple of another row.
10. pivotal row
11. Systems of linear equations can be solve by:

Method 1 back substitution p.6

Method 2 using the elementary row operations on the augmented matrix to reduce it to a strict triangular form p.8 – 9. The elementary row operations to not alter the solution of the system. This method fails if the pivot ends up being 0 at some point, producing an inconsistent system of equations. See new methods in next section.
1.2 **Row Echelon Form**

1. lead variables

2. free variables

3. row echelon form of a matrix p.15. What’s the difference between the strict triangular form and the row echelon form?

4. Gaussian elimination – Method 3 for solving a system of equations using row operations to put a matrix into row echelon form.

5. reduced row echelon form of a matrix p.15. What’s the difference between the strict triangular form, the row echelon form, and the reduced row echelon form?

6. Gauss-Jordan reduction – Method 4 for solving for **undetermined** systems of equations using row operations to put a matrix into reduced row echelon form.

7. **undetermined** systems: fewer equations than unknowns

8. **overdetermined** systems: more equations than unknowns

9. **homogeneous systems** = systems whose right hand side is zero. Homogeneous systems are always consistent (i.e. they will always have solutions):

   (a) If there are the same number of variables as equations, the system has a unique solution: \( x_1 = 0, x_2 = 0, \ldots, x_n = 0. \)

   (b) If there are more variables than equations, then there are infinitely many solutions because of the free variables.
1.3 Matrix Algebra

1. matrix notation: $A = (a_{i,j})$ or $B = (b_{i,j})$

2. vectors = matrices with only one row (row-vector), or only one column (column-vector),

   and they could be equal: $egin{pmatrix} 2 \\ 3 \end{pmatrix} = (2, 3)$

3. $\mathbb{R}^n$ is the euclidean $n$-space whose elements are column vectors (or simply vectors)

4. $i^{th}$ row of a matrix is $a(i,:)$, and $j^{th}$ column of a matrix is $a(:,j)$,

5. $A = B$ if $a_{i,j} = b_{i,j}$ for each $i$ and $j$

6. $\alpha A, A + B, A - B$ versus the product of matrices $AB$ (page 38)

7. linear combination of vectors

8. algebraic rules p 41

9. $AB \neq BA$, i.e. multiplication of matrices is not commutative

10. identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

11. nonsingular matrix (or invertible matrix): if the matrix has an inverse

12. singular matrix: if it is not invertible

13. transpose of a matrix $A^T$ (swap the columns with the rows) and the algebraic rules for the transposes

14. symmetric matrices: if $A^T = A$
1.4 Elementary Matrices

1. Elementary matrices = matrices obtained by performing one elementary row operation on $I$. There are three types:

(Type I) interchanging two rows: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(Type II) constant multiple of a row: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(Type III) adding a multiple of a row to another row: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

2. $E$ is an elementary matrix $\Rightarrow E$ has an inverse $E^{-1}$ which is also an elementary matrix

3. $B$ is row equivalent to $A$ if multiplying $A$ by a series of elementary matrices we get $B$

4. The following are equivalent:
   
   (a) $A$ is nonsingular
   (b) $Ax = 0$ has only one solution: $x = 0$
   (c) $A$ is row equivalent to $I$.

5. $Ax = 0$ if $A$ is nonsingular

6. finding $A^{-1}$ using the elementary row operations (page 66).

7. diagonal matrix: if the entries of the diagonal are the only possible nonzero entries of the matrix

8. upper triangular matrix: if the entries below the diagonal are zero (some of the other entries could be zero as well)

9. lower triangular matrix: if the entries above the diagonal are zero (some of the other entries could be zero as well)

10. triangular matrix: if it is either upper or lower triangular matrix (recall strict triangular matrix).

11. skip Triangular Factorization.