1. (50 points) Sketch the graph of the curve: $f(x) = \frac{x}{(x - 1)^2}$

- Domain of $f$:
  Solution: $\{x|x \neq 1\} = (-\infty, 1) \cup (1, \infty) = \mathbb{R} - \{1\}$.

- y-intercept:
  Solution: letting $x = 0 \implies f(0) = 0$, so $(0, 0)$ is the x-intercept
  Solution: y-intercept: $f(0) = 0$

- Horizontal/Slant Asymptotes:
  Solution: $\lim_{x \to \infty} \frac{x}{2(x - 1)} = 0$, and $\lim_{x \to -\infty} \frac{x}{2(x - 1)} = 0$,
  so $y = 0$ is a HA.

- Vertical Asymptotes:
  Solution: Solve for $x$ in $(x - 1)^2 = 0$, and notice that $\lim_{x \to 1^-} \frac{x}{(x - 1)^2} = \infty$, and $\lim_{x \to 1^+} \frac{x}{(x - 1)^2} = \infty$, $x = 1$ is a VA.

- $f'(x) = \frac{(x - 1)^2(1) - x \cdot (2)(x - 1)}{(x - 1)^4} = \frac{-x^2 + 1}{(x - 1)^4} = \frac{(x - 1)(-x - 1)}{(x - 1)^4} = \frac{-x - 1}{(x - 1)^3}$. 


• Critical points:
  Solution: \( f'(x) = 0 \Rightarrow -x - 1 = 0 \Rightarrow x = -1. \) Also, \( f'(x) \) DNE at \( x = 1. \)

• \( f \) positive/negative:
  Solution: \( f'(x) \) is negative on \((-\infty, -1)\) and \((1, \infty)\), and positive on \((-1, 1)\), so \( f(x) \) is decreasing on \((-\infty, -1)\) and \((1, \infty)\) and increasing on \((-1, 1)\).

• local min/max:
  Solution: Local minimum \( f(-1) = \frac{-1}{4} \), no local maximum.

\[
\begin{align*}
\bullet \quad f''(x) &= \frac{(x-1)^3(-1) - (-x-1)(3)(x-1)^2}{(x-1)^6} \\
&= \frac{2(x+2)}{(x-1)^4}
\end{align*}
\]

• Possible inflection points:
  Solution: \( f''(x) = 0 \Rightarrow 2(x+2) = 0 \Rightarrow x = -2. \) \( f'' \) DNE at \( x = 1. \) Inflection points: \((-2, \frac{-2}{9})\)

• Concavity of \( f \):
  Solution: \( f'' \) is negative on \((-\infty, -2)\), and positive on \((-2, 1)\) and \((1, \infty)\). So \( f \) is concave down on \((-\infty, -2)\), and concave up on \((-2, 1)\) and \((1, \infty)\).

• Graph:

![plot.png]
2. (25 points) Two cars start moving from the same point. One travels south at 30 m/h, and the other travels east at 40 m/h. At what rate is the distance between them changing 1 hour later?
Solution:

\[ \begin{align*}
\frac{dx}{dt} &= 30 \text{ m/h} \\
\frac{dy}{dt} &= 40 \text{ m/h}
\end{align*} \]

Thus, when \( t = 1 \) : \( x = 30 \text{ m} \) and \( y = 40 \text{ m} \). Since \( x^2 + y^2 = z^2 \), it follows that \( z = \sqrt{30^2 + 40^2} = 50 \).

Since \( x^2 + y^2 = z^2 \), we have that

\[ \frac{2x}{dt} \frac{dx}{dt} + 2y \frac{dy}{dt} = \frac{dz}{dt} \]

so

\[ \frac{dz}{dt} = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{z} = \frac{30 \cdot 30 + 40 \cdot 40}{50} = 50 \text{ m/h}. \]

3. (25 points) Evaluate the limit (show your work): \( \lim_{x \to 0^+} x^{x^3} \)

Solution: Solution: Let \( x^{x^3} = y \). Then \( x^3 \ln x = \ln y \), so

\[ \begin{align*}
\lim_{x \to 0^+} \ln y &= \lim_{x \to 0^+} x^3 \ln x = \lim_{x \to 0^+} \frac{\ln x}{x^{-3}} = \lim_{x \to 0^+} \frac{1}{(-3)x^{-4}} = \lim_{x \to 0^+} \frac{x^4}{-3x} = \lim_{x \to 0^+} \frac{x^3}{-3} = 0.
\end{align*} \]

Thus \( \lim_{x \to 0^+} y = e^{\lim_{x \to 0^+} \ln y} = e^0 = 1. \)