Show all necessary work in each problem to receive credit.

1. (5 points) A relation is defined on \( \mathbb{Z} \) by \( xRy \) if \( 3 \mid (4x - y) \). Show that \( R \) is transitive.

Proof: Let \( x, y, z \in \mathbb{Z} \) such that \( xRy \) and \( yRz \). Then \( 3 \mid (4x - y) \) and \( 3 \mid (4y - z) \), which implies that \( 4x - y = 3k \) and \( 4y - z = 3\ell \), for some \( k, \ell \in \mathbb{Z} \). We wish to show that \( 3 \mid (4x - z) \).

Consider \( 4x - z = (4x - y) + (4y - z) + (-3y) = 3k + 3\ell - 3y = 3(k + \ell - y) \). Since \( k + \ell - y \in \mathbb{Z} \), it follows that \( 3 \mid (4x - z) \).

2. (4 points) Let the relation \( R \) be an equivalence relation on \( \mathbb{Z} \), where \( xRy \) if \( x^2 = y^2 \). Find the equivalence classes (note that you don’t have to prove that \( R \) is an equivalence relation anymore.)

Solution:

\[
[0] = \{0\} \\
[1] = \{\pm 1\} \\
[2] = \{\pm 2\} \\
\vdots
\]

This generalizes to \( [x] = \{\pm x\} \), for all \( x \in \mathbb{Z} \), with \( [0] = \{0\} \).
3. (5 points) Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ by $f(x) = 2x + 1$.

1. What is the range of $f$?

2. Is $f(x)$ one-to-one.? (If it is, then prove it, if it isn’t, then provide a counterexample)

Solution:

1. Range of $f$ is $\text{Range} = \{2k + 1 : k \in \mathbb{Z}\}$, i.e. all the odd integers.

2. The function is one-to-one.

   Proof: Let $a, b \in \mathbb{Z}$. Assume that $f(a) = f(b)$, so $5a - 7 = 5b - 7$. Adding 7 to both sides we get $5a = 5b$ and so $a = b$. Therefore, $f$ is one-to-one.

3. (6 points) Let the relation $R$ be defined on \{a, b, c, d\} by $R = \{(a, a), (a, d), (d, a)\}$.

1. Is $R$ a function? Why?

2. Is $R$ an equivalence relation? Why?

Solution:

1. $R$ is not a function since the element $a$ gets mapped to two different elements (or also, since $d$ does not get mapped to any element).

2. $R$ is not an equivalence relation since it is not reflexive (there is no pair $(b, b)$). Also, it is not transitive (since we have the pairs $(d, a)$ and $(a, d)$, we should have the pair $(d, d)$).