1 Introduction to Counting

1.1 Introduction

In this chapter you will learn the fundamentals of enumerative combinatorics, the branch of mathematics concerned with counting. While enumeration problems can be arbitrarily difficult, the fundamentals are easy to master and will prepare you for the more difficult problems ahead. You will be surprised at the degree of difficulty of the problems that you can solve using only these simple tools. Our goal is that this chapter be sufficient to meet the needs of students in MA1025, but some of you might want additional reading material or additional exercises. There are currently two copies of *Discrete Mathematics and Its Applications*, by Kenneth Rosen, on two-hour reserve in the library for the students in MA2025. By now, all of those students have their own copies, so it should be available. If you want more to read, and more problems to work, it is a good choice. The material in this chapter corresponds to Section 5.1, and Section 5.2 through page 335. A number of problems can be found at the end of this chapter. You will also find a list of suggested exercises from the Rosen text.

Upon completion of this chapter, you will have mastered the following:

- The Sum Rule
- The Product Rule
- The basic Inclusion-Exclusion Principle

You will also encounter a device called a tree diagram and a tool colloquially known as the Pigeonhole Principle, but the first is infrequently used and the second shows up in some very difficult problems, so I don’t think mastery of those tools is the goal.

1.2 Overview and Definitions

The *sum rule* tells us in how many ways one can make a single choice from two disjoint sets of alternatives. The *product rule* tells us in how many ways one can make one choice from each of two sets of alternatives. Both rules generalize to larger families of sets. The basic *Principle of Inclusion-Exclusion* extends the sum rule to situations in which the two sets of alternatives are not disjoint. This, too, generalizes to larger collections of sets. A finite sequence of choices can be represented by a *tree diagram*, in which the root represents the initial state, leaves represent outcomes, internal vertices represent intermediate states, and edges represent choices. The Pigeonhole Principle, in its picturesque form, says that if $k + 1$ pigeons fly into $k$ pigeonholes, at least one pigeonhole must contain at least two pigeons.
1.3 The Sum and Product Rules

The most fundamental rules are the Sum Rule and the Product Rule.

**Theorem 1** *(Sum Rule)* If \( A \cap B = \emptyset \), then \(|A \cup B| = |A| + |B|\).

Although the sum rule tells us that the cardinality of the union of two disjoint sets is the sum of the cardinalities of the two sets, it is typically applied to problems that do not immediately remind us of sets.

**Example 1**: Suppose that you are in a restaurant, and are going to have either soup or salad but not both. There are two soups and four salads on the menu. How many choices do you have? By the Sum Rule, you have \(2 + 4 = 6\) choices.

**Theorem 2** *(Product Rule)* For any choice of sets \( A \) and \( B \), \(|A \times B| = |A||B|\).

The product rule tells us that the cardinality of the Cartesian Product of two sets is the product of the cardinality of the two sets. Back in that same restaurant, there are \(2 \cdot 4 = 8\) ways to have both soup and salad. Both rules generalize to larger numbers of sets, although the generalization of the sum rule requires that the sets in question are pairwise disjoint.

**Example 2**: The first two of these problems apply the generalized Product Rule. We begin with a couple of definitions: An *alphabet* is a finite set of symbols. A *string* of length \( k \) over an alphabet \( A \) is a finite sequence \( a_1a_2...a_k \) of symbols from \( A \), with repetition allowed.

(a) How many distinguishable strings of length 3 over the alphabet \( \{A, B, \ldots, Z\} \) exist?

**Solution**: Since there are 26 choices for the first symbol, 26 for the second, and 26 for the third, then by the Product Rule there are \(26^3 = 17576\) such strings of length 3.

(b) How many strings of length 4 over the alphabet \( \{0, 1, \ldots, 9\} \) do not begin with 0?

**Solution**: There are nine choices for the first symbol, and ten for each of the second, third, and fourth. By the Product Rule, there are \(9 \cdot 10^3 = 9000\) strings of length 4 that do not begin with 0. (Note: this is also the number of 4-digit positive integers with no leading zeros.)

(c) The standard California license-plate number begins with a nonzero decimal digit that is followed by three uppercase alpha characters, which are in turn followed by three decimal digits. How many of these license numbers exist?

**Solution**: By the preceding problems and the basic Product Rule, there are

\[9 \cdot 26^3 \cdot 10^3 = 158184000\]

such numbers. The fact that we are inserting the string of alpha characters between the first and second digits of the string of decimal digits has no effect on the count.
It’s hard to argue that these are anything but the simplest of rules. By using these rules in combination, you can do quite a bit, but you should expect to find the degree of difficulty rising. Sometimes the added difficulty is simply a matter of bookkeeping, but it can also take the form of additional subtlety.

Example 3: We combine the sum and product rules, and introduce a new tool, to find the number of passwords adhering to some simple constraints. The length must be at least 5 and at most 7. The password must be constructed of uppercase alpha characters and decimal digits, and must contain at least one digit. By the sum rule, the total number \( P \) is given by \( P = P_5 + P_6 + P_7 \), where \( P_i \) is the number of legal passwords of length \( i \). But what is \( P_5 \)? The number of digits in a legal password of length 5 could be as small as 1 or as large as 5, so it appears that we must compute five numbers and find their sum. But notice that the number of illegal passwords of length 5 is easy to count: there are \( 26^5 \) of these. It follows that \( P_5 = 36^5 - 26^5 = 48584800 \). This is an example of indirect counting: to find the number of ways to perform a task in the presence of constraints, we instead count the number of ways to perform the task with no constraints and subtract from it the number of ways to perform the task while violating those constraints. This method is sometimes easier, and should not be overlooked. We can use the same approach to find \( P_6 = 36^6 - 26^6 = 1867866560 \) and \( P_7 = 36^7 - 26^7 = 70332353920 \), and the problem is solved: there are

\[
P_5 + P_6 + P_7 = 72248805280
\]

acceptable passwords.

Example 4:

(a) How many integers \( x \) with \( 1 \leq x \leq 11 \) are divisible by 2?

Solution: \( \lfloor \frac{11}{2} \rfloor = 5 \).

(b) How many integers \( x \) with \( 1 \leq x \leq 11 \) are not divisible by 2?

Solution: We use indirect counting by subtracting those that are divisible by 2 from total number of integers: \( 11 - \lfloor \frac{11}{2} \rfloor = 6 \).

1.4 The Principle of Inclusion-Exclusion

The Principle of Inclusion-Exclusion is the principle that lets us generalize the Sum Rule by counting unions of sets that are not necessarily pairwise disjoint. The basic instance of the principle applies to unions of two sets. The idea is simple: we already know that if two sets \( A \) and \( B \) are disjoint, the cardinality of their union is simply \( |A \cup B| = |A| + |B| \). Suppose, though, that \( A \cap B \neq \emptyset \). If \( x \in A \cap B \), then \( x \) is counted twice in \( |A| + |B| \): once in \( |A| \) and
once in $|B|$. This applies to every element in $A \cap B$, so we must subtract $|A \cap B|$ to correct the overcount. The principle, then, tells us that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$ 

**Example 5:** The following four problems apply the principle of inclusion-exclusion.

(a) Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7\}$. Then $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$. By inspection, $|A \cup B| = 7$, but you can also verify that

$$|A \cup B| = 7 = 5 + 5 - 3 = |A| + |B| - |A \cap B|.$$ 

(b) How many positive integers not bigger than 20 are divisible by either 2 or 3?

**Solution:** There are $\lfloor 20/2 \rfloor = 10$ that are divisible by 2, and $\lfloor 20/3 \rfloor = 6$ that are divisible by 3. But there are also $\lfloor 20/6 \rfloor = 3$ that are divisible by both 2 and 3, so the total is $10 + 6 - 3 = 13$.

(c) How many bitstrings of length eight either begin with 00 or end with 101?

**Solution:** There are $2^6$ that begin with 00, $2^5$ that end with 101, and $2^3$ that start with 00 and end with 101. So the number of bitstrings with at least one of the two properties is $2^6 + 2^5 - 2^3 = 88$.

You can probably convince yourself that the inclusion-exclusion formula for the cardinality of the union of three sets is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$ 

The principle generalizes to more than three sets, and there is an inductive proof of that generalization.

### 1.5 Tree Diagrams

A **tree diagram** offers a way to enumerate outcomes resulting from a finite (and preferably small) sequence of choices. Each vertex represents a state, with the initial state represented by the root and the outcomes represented by the leaves. Edges tending downward from a vertex represent choices. Figure 1 shows a decision tree used to enumerate bitstrings of length three that do not contain 11. The leaves, from left to right, represent the strings 101, 100, 010, 001, and 000. You can see that any branch containing two consecutive 1s has been pruned out, leaving only those that do not contain 11. And you can probably guess that the utility of tree diagrams, like that of truth tables and Venn diagrams, is limited to small problems. On the other hand, the tree structure lends itself to computation, so you will probably see this again.
Figure 1: A decision tree for counting 11-free strings

1.6 The Pigeonhole Principle

The Pigeonhole Principle has as picturesque a statement as any in combinatorics, and in its most basic form the proof is so intuitive as to be obvious. But there are many forms to this principle, and putting it to use can be arduous. Our focus is on relatively easy applications of the principle. Here it is in its simple form:

Theorem 3 (The Simplified Pigeonhole Principle) If \( k \) pigeons occupy \( k - 1 \) pigeonholes, then at least one pigeonhole contains at least two pigeons.

A more useful presentation is the following:

Theorem 4 (The Pigeonhole Principle) If \( k \) pigeons occupy \( j < k \) pigeonholes, then at least one pigeonhole contains at least two pigeons.

This can be restated in terms of functions: If \( A \) and \( B \) are sets, and if \( |A| > |B| \), then there can be no one-to-one mapping \( f : A \to B \). The proof is straightforward. In applying the principle, the difficulty (when there is one) is in deciding what constitutes a pigeonhole and what constitutes a pigeon.

Example 6:

(a) The “hello, world” problem for the pigeonhole principle is the “sock problem”: In your dresser drawer you have a jumble of socks in two colors, say blue and gray. It’s dark, and you don’t want to wake your spouse. How many socks must you grab to guarantee that you have a pair of the same color?

Solution: Three socks suffice. You might end up with three blue, or three gray, but with only two colors you’re guaranteed to have at least two blue or at least two gray. That is, the pigeonholes are the colors, the pigeons are the socks. As you draw one sock from the drawer, you put it in the respective pigeonhole, and one pigeonhole will have at least two socks.
(b) Show that in a group of eight people there must be two whose birthdays fall on the same day of the week.

**Solution:** The pigeons are now the people in the group, and the pigeonholes are the days of the week.

(c) Show that in a group of ten people there must be two whose birthdays fall on the same day of the week.

**Solution:** The pigeons are now the people in the group, and the pigeonholes are the days of the week.

(d) Show that if five integers are selected from the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, there must be two whose sum is 9.

**Solution:** Consider the four pairs $\{1, 8\}, \{2, 7\}, \{3, 6\}$, and $\{4, 5\}$. Those are the pigeonholes. The pigeons are the five selected integers. Two must be elements of the same pigeonhole.

A generalization of the principle is this:

**Theorem 5** If $n$ pigeons occupy $k$ pigeonholes, then at least one pigeonhole contains at least $\lceil n/k \rceil$ pigeons.

We can use this version to answer more difficult questions. What is the smallest $n$ such that at least one of $k$ boxes must contain at least $r$ of $n$ objects? By Theorem 4, we need $\lceil n/k \rceil \geq r$. So the smallest integer $n$ that forces some box to contain $r$ of $n$ objects is $n = k(r - 1) + 1$.

How can you avoid having at least $r$ objects in some box? By setting $n < r$, of course. But by setting $n \leq k(r - 1)$, you stand a chance at having fewer than $r$ objects in every box.

**Example 7:**

(a) In your dresser drawer you have a jumble of socks in two colors, say blue and gray. It’s dark, and you don’t want to wake your spouse. How many socks must you grab to guarantee that you have 3 socks of the same color?

**Solution:** Five socks suffice. You might end up with more than three blue, or three gray, but with only two colors you’re guaranteed to have at least three blue or at least three gray. That is, the pigeonholes are the colors, the the pigeons are the socks. As you draw one sock from the drawer, you put it in the respective pigeonhole, and one pigeonhole will have at least three socks.

(b) Show that in a group of 25 people there must be four whose birthdays fall on the same day of the week.

**Solution:** The pigeons are now the people in the group, and the pigeonholes are the days of the week. And so $\lceil \frac{25}{7} \rceil = 4$ people share the same day of the week.
1.6.1 Exercises

Here are some exercises that apply the techniques presented in this chapter.

1. A store has t-shirts in 6 different styles and 4 different sizes. How many different kinds of t-shirts does the store have?

2. In Example 2, we described the standard California license-plate number. How many such numbers have
   (a) no repeated digit?
   (b) no repeated letter?
   (c) no repeated symbol?

3. In how many ways can a ballot be validly marked if a person is to vote on three questions, and at least one question must be answered (i.e. one or two questions may be skipped, but not all three of them), if for question 1 there are four options to choose from, for question 2 there are two options to choose from, and two more options to chose from for question 3?

4. How many integers $n$, with $1 \leq n \leq 200$, are not divisible by 2, 3, or 5?

5. Given five points in the plane, with integer coordinates, prove that there are two with the property that the midpoint of the line segment joining them has integer coordinates.

6. Let $S$ be a six-element subset of $\{1, 2, \ldots, 14\}$. Show that there are two proper nonempty subsets of $S$, say $A$ and $B$ such that the sum of the values of the elements of $A$ is the same as the sum of the values of the elements of $B$. Recall that $A$ is a proper subset of $B$ if $A \subset B$.

7. Use the principle of mathematical induction to generalize the product rule to arbitrarily many tasks.

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Solutions to Exercises, Chapter 1 of Combinatorics Notes for MA1025

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1. A store has t-shirts in 6 different styles and 4 different sizes. How many different kinds of t-shirts does the store have?

   **Solution:** There are $6 + 6 + 6 + 6 = 6 \times 4 = 24$ (6 choices for each size).

2. In Example 2, we described the standard California license-plate number. How many such numbers have
(a) no repeated digit?
(b) no repeated letter?
(c) no repeated symbol?

Solution:

(a) There are still $26^3$ ways to choose the alpha characters, but now there are 9 choices for the first digit, 9 for the second, 8 for the third, and 7 for the fourth. In all, there are $9 \cdot 26^3 \cdot 9 \cdot 8 \cdot 7$ such numbers.

(b) There are $9 \cdot 10^3$ ways to choose the digits, but now there are $26 \cdot 25 \cdot 24$ ways to choose the letters. In all, there are $9 \cdot 26 \cdot 25 \cdot 24 \cdot 10^3$ such numbers.

(c) Combining the results from (a) and (b), there are $9 \cdot 26 \cdot 25 \cdot 24 \cdot 9 \cdot 8 \cdot 7$ license numbers with no repeated symbols.

3. In how many ways can a ballot be validly marked if a person is to vote on three questions, and at least one question must be answered (i.e. one or two questions may be skipped, but not all three of them), if for question 1 there are four options to choose from, for question 2 there are two options to choose from, and two more options to chose from for question 3?

Solution: There are 5 ways to answer question 1 (one of the 4 options, or skipping that question). Then there are 3 ways to answer each of the questions 2 and 3. However we counted the option that no question was answered which is not a valid ballot. Thus the are $5 \cdot 3 \cdot 3 - 1 = 53$ different possible valid ballots.

4. How many integers $n$, with $1 \leq n \leq 200$, are not divisible by 2, 3, or 5?

Solution: The number of integers $n$, with $1 \leq n \leq 200$, that are divisible by at least one of 2, 3, or 5 is

$$\left\lfloor \frac{200}{2} \right\rfloor + \left\lfloor \frac{200}{3} \right\rfloor + \left\lfloor \frac{200}{5} \right\rfloor - \left\lfloor \frac{200}{6} \right\rfloor - \left\lfloor \frac{200}{10} \right\rfloor - \left\lfloor \frac{200}{15} \right\rfloor + \left\lfloor \frac{200}{30} \right\rfloor$$

$$= 100 + 66 + 40 - 33 - 20 - 13 + 6 = 146,$$

so the number of integers in $\{1, 2, \ldots, 200\}$ divisible by none of them is $200 - 146 = 54$.

5. Given five points in the plane, with integer coordinates, prove that there are two with the property that the midpoint of the line segment joining them has integer coordinates.

Proof: Given two integer coordinate pairs $(x_1, y_1)$ and $(x_2, y_2)$, the midpoint of the line segment joining them has coordinates $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$, which is an integer pair only if both $x_1 + x_2$ and $y_1 + y_2$ are even, which requires that the parity of $x_1$ agrees with that of $x_2$ and that the parity of $y_1$ agrees with that of $y_2$. There are four “parity patterns” that an ordered pair might have. These are (even,even), (odd,odd), (even,odd), and (odd,even). Given five ordered pairs, two must share one of these patterns, but then the midpoint of the line segment joining them has integer coordinates. ■
6. Let $S$ be a six-element subset of $\{1, 2, \ldots, 14\}$. Prove that there are two proper nonempty subsets of $S$, say $A$ and $B$ such that the sum of the values of the elements of $A$ is the same as the sum of the values of the elements of $B$. Recall that $A$ is a proper subset of $B$ if $A \subset B$.

**Proof:** There are $2^6 - 2 = 62$ nonempty proper subsets of $S$. The sum of the elements of each such subset lies between 1 and $14 + 13 + 12 + 11 + 10 = 60$. By the pigeonhole principle, there must be a pair of subsets with identical sums. ■

7. Use the principle of mathematical induction to generalize the product rule to arbitrarily many tasks.

**Solution:** The base case is the case of two tasks, which is covered by the basic product rule. Assume that $k$ tasks, where the $i$th task can be performed in $n_i$ ways, can be performed in $\prod_{i=1}^{k} n_i$ ways. Suppose we must perform $k + 1$ tasks, with $n_i$ the number of ways in which the $i$th task can be performed. By the induction hypothesis, there are $\prod_{i=1}^{k} n_i$ ways to perform the first $k$ tasks, and by hypothesis there are $n_{k+1}$ ways to perform the $(k+1)$st task. By the basic product rule, there are $\left(\prod_{i=1}^{k} n_i\right) \cdot n_{k+1} = \prod_{i=1}^{k+1} n_i$ ways to perform all $k + 1$ tasks. ■

If you want additional exercises, here are some suggestions for exercises from the text by Rosen. At the end of Section 5.1, you might work problems 3 – 15 (odds), 21, 26, 41, 43. In Section 5.2, you might work problems 3, 5, 9, 13, 19, 29.

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1.7 Assessment

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Self-Quiz on Fundamentals of Counting

1. Bud’s Grill offers an earlybird dinner special, which includes one of four entrées, either a soup (two kinds) or a salad (four choices), and a dessert (three choices). In how many different ways can one order all three courses from the special menu?

2. How many functions are there from a 5-element set $A$ to a 6-element set $B$? How many are one-to-one?

3. How many subsets of $\{1, 2, \ldots, 8\}$ contain more than one element?

4. How many bitstrings of length eight either start with 01 or end with 01?

5. How many numbers must be selected from $\{1, 2, \ldots, 10\}$ to guarantee that there is a pair whose sum is 11? How many, if the sum is to be 13?
6. At a party, there are $n$ people for some $n \geq 2$. Show that there must be two people at the party who know precisely the same number of other people at the party.

Self-Quiz Solutions: Fundamentals of Counting

1. Bud’s Grill offers an earlybird dinner special, which includes one of four entrees, either a soup (two kinds) or a salad (four choices), and a dessert (three choices). In how many different ways can one order all three courses from the special menu?

Solution: A hypothetical diner can order an entree in four ways, one of six first courses, and one of three desserts. By the product rule, there are $4 \cdot 6 \cdot 3 = 72$ ways to order.

2. How many functions are there from a 5-element set $A$ to a 6-element set $B$? How many are one-to-one?

Solution: Let $A = \{a_1, a_2, \ldots, a_5\}$. In the first case, there are 6 choices for the image of each element $a_i \in A$, so by the product rule there are $6^5 = 7776$ such functions. If the function must be one-to-one, there are six choices for the image of $a_1$, then five choices for the image of $a_2$, etc. So there are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 6! = 720$ one-to-one functions from $A$ to $B$.

3. How many subsets of $\{1, 2, \ldots, 8\}$ contain more than one element?

Solution: All except the empty set and the eight singletons, so altogether $2^8 - 9 = 247$ subsets contain more than one element.

4. How many bitstrings of length 8 either start with 01 or end with 01?

Solution: There are $2^6$ that start with 01, $2^6$ that end with 01, and $2^4$ with both properties, so the number in question is $2 \cdot 2^6 - 2^4 = 112$.

5. How many numbers must be selected from $\{1, 2, \ldots, 10\}$ to guarantee that there is a pair whose sum is 11? How many, if the sum is to be 13?

Solution: There are five pairs ($\{1, 10\}$, $\{2, 9\}$, $\{3, 8\}$, $\{4, 7\}$, and $\{5, 6\}$) that add up to 11, so we must choose six numbers to guarantee that we have both elements in one of the five sets. There are four pairs ($\{3, 10\}$, $\{4, 9\}$, $\{5, 8\}$, and $\{6, 7\}$) that add up to 13, and there are also the numbers 1 and 2, neither of which combines with another element of the set to make 13. So we must choose seven numbers to guarantee that two of them add up to 13.

6. At a party, there are $n$ people for some $n \geq 2$. Show that there must be two people at the party who know precisely the same number of other people at the party.

Proof: The number of people known by each celebrant (these are the pigeons) is an integer between 0 and $n - 1$ (these are the labels on the pigeonholes). So it appears that we have ten pigeonholes for ten pigeons. But now there are two cases. If any
individual knows nobody, then the pigeonhole labeled \( n - 1 \) must be empty. Otherwise, the pigeonhole labeled 0 must be empty. In either case, we have \( n \) pigeons but only \( n - 1 \) pigeonholes. One of them must contain at least two pigeons. \( \blacksquare \)