1 Chapter 1: SETS

A set is a collection of objects. The objects of the set are called elements or members. Use capital letters: A, B, C, S, X, Y to denote the sets. Use lower case letters to denote the elements: a, b, c, x, y.

If \( x \) is an element of the set \( X \), we write \( x \in X \). If \( x \) is not an element of the set \( X \), we write \( x \not\in X \).

1.1 Describing a set

1. List all elements if the set consists of a small number of elements:
   \[ X = \{a, b, c\} \]
   \[ A = \{1, 2, \ldots, 100\} \] – need to list the first two elements to if they are consecutive
   \[ S = \{1, 3, 5, \ldots\} \] – list the first 3 elements to give away the pattern. (Not correct to list: \( S = \{1, 3, 5, 7, \ldots\} \), which is redundant, nor \( S = \{1, 3, \ldots\} \) because of not enough information.)
   NOTE:
   \[ A = \{1, 2, 3\} = \{2, 1, 3\} = \{1, 1, 3, 2\} \]
   \[ \emptyset = \{\} \] is the empty set
   \[ \{\emptyset\} \neq \emptyset \] as the first set has one element, and the second set has no elements.

2. A set with condition(s): \( S = \{x \mid p(x)\} \) or \( \{x : p(x)\} \), that is: \( S \) contains all the elements \( x \) that satisfy the condition (or have the property) \( p(x) \), where \( p(x) \) a property that depends on \( x \).
   Ex: \( A = \{x : x \text{ is even}\} \)  
   \[ S = \{x : (x - 1)(x + 2) = 0\} \]
   \[ T = \{x : |x| = 1\} \]
   \[ X = \{x : x \text{ is a student in MA1025}\} \].
   A more complex example: Let \( A = \{1, 2, \ldots, 10\} \). Then define \( B = \{x \in A : x < 7\} = \{1, 2, 3, 4, 5, 6\} \)

For a set \( S \), we write \(|S|\) to denote the number of elements in the set \( S \). The number \(|S|\) is called the cardinality of \( S \).

Ex: for the examples above: \(|S| = 2, |\{4\}| = 1, |\emptyset| = 0, |\{\emptyset\}| = 1\).
**Special sets**

\[ N = \{1, 2, \ldots, \} \] is the set of all positive whole numbers

\[ Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \] is the set of integers (whole numbers)

\[ E = \{y : y \text{ is an even integer }\} = \{\ldots, -4, -2, 0, 2, 4, \ldots\} \]

\[ Q = \{\frac{p}{q} : p, q \in Z, q \neq 0\} \] is the set of the rational numbers

\[ I = \text{the set of irrationals}, \text{ for example: } \pi, \sqrt{2}, -\sqrt{3} \]

\[ R = \text{the real numbers} \]

\[ R^+ = \text{the positive real numbers} \]

\[ C = \text{the set of complex numbers}: a + bi \]

### 1.2 Subsets

A set \( A \) is a subset of a set \( B \) if every element of \( A \) is an element of \( B \). If \( A \) is a subset of \( B \), we write \( A \subseteq B \).

\[ \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C} \]

If a set \( A \) is not a subset of a set \( B \), we write \( A \nsubseteq B \). In this case, there is an element in the set \( A \) that is not in \( B \).

The empty set \( \emptyset \) is a subset of every set.

**INTERVALS:** Let \( a, b \in \mathbb{R} \)

\[ [a, b] = \{x \in \mathbb{R} : a \leq x \leq b\} \]

\[ (a, b) = \{x \in \mathbb{R} : a < x < b\} \]

\[ (a, \infty) = \{x \in \mathbb{R} : a < x\} \]

\[ (-\infty, b] = \{x \in \mathbb{R} : x \leq b\} \]

Two sets \( A \) and \( B \) are equal, written as \( A = B \), if they have exactly the same elements. By definition \( A = B \) means \( A \subseteq B \) and \( B \subseteq A \).

If they are not equal then we write \( A \neq B \).

A set \( A \) is a proper subset of a set \( B \) if \( A \subseteq B \) and \( A \neq B \), and it is denoted by \( A \subset B \).

Ex: Let \( A = \{a\} \) and \( B = \{a, b\} \).

When we talk about subsets, we are concerned with subsets of a larger set, usually called universal set, denoted by \( U \).

We can use Venn Diagrams to represent a set:
From the diagram: \( x \in A, y \in B, z \in A, z \in B, w \notin A, w \notin B. \)

For a set \( A \), the power set \( \mathcal{P}(A) \) of \( A \) is the set of all subsets of \( A \).

Ex: \( A = \{a, b\} \). Then \( \mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \neq \{\emptyset, a, b, \{a, b\}\} \) since \( a, b \) are not sets without the curly brackets. \( \mathcal{P}(A) = 2^2 = 2^{|A|} \).

In general: \(|\mathcal{P}(A)| = 2^{|A|}\).

1.3 Set Operations

Let \( A \) and \( B \) be two sets. The following are ways of combining two or more sets:

1. The \textbf{intersection} of \( A \) and \( B \): \( A \cap B = \{x : x \in A \text{ and } x \in B\} \). If \( A \cap B = \emptyset \), then \( A \) and \( B \) are disjoint.

2. The \textbf{union} of \( A \) and \( B \): \( A \cup B = \{x : x \in A \text{ or } x \in B\} \).

3. The \textbf{difference} of \( A \) and \( B \): \( A \setminus B = \{x : x \in A \text{ and } x \notin B\} \).
4. The complement of $A$: $\bar{A} = \{x : x \notin A\} = U \setminus A$, where $U$ is the universal set.

5. The relative complement of $B$ in $A$ (same as the difference of the two sets):
   $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.

1.4 Indexed Collection of Sets

Suppose that $A_1, A_2, \ldots, A_n$ is a collection of collection of sets, ($n \geq 3$). The following are definitions of ways of combining two or more sets:

1. The intersection of the $n$ sets $A_1, A_2, \ldots, A_n$ is:

$$\bigcap_{i=1}^{n} A_i = \{x : x \in A_i, \text{for all values } i, 1 \leq i \leq n\}.$$

2. The union of the $n$ sets $A_1, A_2, \ldots, A_n$ is:
\[
\bigcup_{i=1}^{n} A_i = \{x : x \in A_i, \text{ for at least one value } i, 1 \leq i \leq n\}.
\]

Example: Let \(A_i = \{i, i+1\}, 1 \leq i \leq 10\). What are the intersection and the union of them all?

1. \(\bigcap_{i=1}^{10} A_i = \emptyset\).
2. \(\bigcup_{i=1}^{10} A_i = \{1, 2, \ldots, 11\}\).

Note: If we have different index sets, we have different results:

Let \(A_i = \{i, i+1\}\), and the index set \(I = \{1, 5, 10\}\). Then

1. \(\bigcap_{i \in I} A_i = \emptyset\).
2. \(\bigcup_{i \in I} A_i = \{1, 2, 5, 6, 10, 11\}\).

Let \(I\) be the indexed set. For each \(\alpha \in I\) there is a set, which is deonoted by \(S_\alpha\) that corresponds to \(\alpha\). Hence

\[\{S_\alpha\}_{\alpha \in I}\]

is called an indexed collection of sets. We then have

\[\bigcap_{\alpha \in I} S_\alpha \cup \bigcup_{\alpha \in I} S_\alpha.\]

Example 1: Let \(A_n = [-\frac{1}{n}, \frac{1}{n}] = \{x : -\frac{1}{n} \leq x \leq \frac{1}{n}\}\), and let \(I = \mathbb{N}\). What is their union and intersection?

\[\bigcup_{n \in \mathbb{N}} A_n = [-1, 1], \bigcap_{n \in \mathbb{N}} A_n = \{0\}.\]

Example 2: Let \(S_n = \{n, 2n\}\), and let \(I = \{1, 2, 3, 4\}\). What are \(S_n\) for \(n = 1, 2, 3, 4\)? What is their union and intersection?

\(S_1 = \{1, 2\}, S_2 = \{2, 4\}, S_3 = \{3, 6\}, S_4 = \{4, 8\}\).

\[\bigcup_{n \in \mathbb{I}} S_n = \{1, 2, 4, 6, 8\}, \bigcap_{n \in \mathbb{I}} S_n = \emptyset.\]
1.5 Partitions of Sets

Let $S$ be a set of subsets of a set $A$. Ex: $A = \{a, b, c, d\}$ and $S = \{\{a\}, \{b, c\}, \{d\}\}$. Then $S$ is pairwise disjoint if every two distinct subsets in (that belong to) $S$ are disjoint. If $S$ is pairwise disjoint set of nonempty subsets of $A$ such that every element of $A$ belongs to some subset of $S$, then $S$ is called a partition of $A$.

Ex: Let $A = \{1, 2, \ldots, 7\}$. Which of the following are partitions?

- $S_0 = \{\{1\}, \{1, 2, 3\}, \{4, 5, 6, 7\}\} – No$
- $S_1 = \{\{1\}, \{2, 3\}, \{4, 5, 6\}, \{7\}\} – Yes$
- $S_2 = \{\{1\}, \{1, 2, 3\}, \{5, 6\}\} – No$
- $S_3 = \{\emptyset, \{1\}, \{2, 3, 4, 5, 6\}\} – No$

That is: $S$ is a partition if:

1. if $X \in S$, then $X \neq \emptyset$,
2. if $X, Y \in S$, then $X \cap Y = \emptyset$,
3. $\cup_{X \in S} X = A$.

Another example: $S = \{\{1, 2, 5\}, \{3, 4\}, \{6, 7\}\}$. 

Figure 4: A partition of the set $A$
1.6 Cartesian Product

The Cartesian product of two sets $A$ and $B$ is

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Note that $(a, b)$ is an ordered pair, so $(a, b) \neq (b, a)$ (versus the set $\{a, b\} = \{b, a\}$)

Example: Let $A = \{x, y\}$ and $B = \{1, \{2\}, 3\}$. Then

$$A \times B = \{(x, 1), (x, \{2\}), (x, 3), (y, 1), (y, \{2\}), (y, 3)\}$$

$$B \times A = \{(1, x), (1, y), (\{2\}, x), (\{2\}, y), (3, x), (3, y)\}.$$

Note that $A \times B \neq B \times A$.

What is $|A \times B|$? $|A \times B| = |A| \times |B| = 6$ in this case.

If $A = \emptyset$, then $A \times B = \emptyset$ and $B \times A = \emptyset$, for any set $B$, by definition.