Some of My Unsolved Problems in Graph Theory

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Topics

• Rational Distances
• Chromatic Number of the Plane
• Crossing Number

• Square of an Oriented Graph

• Cycle Modularity
• Chords of a Cycle
What is the largest possible subset of points in the plane with no three of the points collinear and no four of the points concyclic, and having all interpoint distances rational?
Rational Distances for Points on a Circle

(-7/10, 12/5)  (7/10, 12/5)
(-5/2, 0)  (5/2, 0)
(-7/10, -12/5)  (7/10, -12/5)
Circles

It’s impossible to find an infinite set of points on the circle $x^2 + y^2 = r^2$ where the distance between any pair of them is rational. However, there are arbitrarily many of them.
How many points can be found on the (half) parabola $y = x^2$, $x \geq 0$, so that the distance between any pair of them is rational?
Points on a Parabola

\[ y = x^2 \]
Proposition: There are infinitely many rational distance sets of 3 point on the parabola $y = x^2$.

Proof:
Pick two rational numbers, say 1 and 3, and let those be the lengths of the segments AB and BC. Fix these distances and slide the points up the parabola. The distance AC will increase, bounded above by 4. Since the rationals are dense in the reals, there are many placements of the points where AC has rational length. Hence, there are infinitely many such triples.
More Recent Results

• Avion Braveman (2001) and others produced three point solutions.

• Campbell (1997) used elliptic curves to compute a four point solution and proved that there are infinitely many such sets.
Ellipses?
Hyperbolas?
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Chromatic Number of the Plane $\chi_P$

- Let $G$ be the graph in the plane where $V = \mathbb{R}^2$ and two points are joined by an edge when their Euclidean distance is 1.
- Define $\chi_P = \chi(G)$
- **Hadwiger-Nelson** (~1950): $\chi_P = ?$
- Any subgraph of $G$ is called a *unit distance graph*.
$4 \leq \chi_P$ (The Moser Spindle)
$\chi_P \leq 7$ (Hexagonal Tiling)
\( \chi_P \leq 6 \) ?
(Pritikin - 1998)

Every UDG of order \( \leq 6197 \) is 6-colorable.

Every UDG of order \( \leq 12 \) is 4-colorable.
Some Known UDGs

- Petersen Graph
- Cycle $C_n$
- Grid Graph
A UDG is maximal if the addition of any edge results in a graph which is not a UDG.

**Not Maximal**

![Not Maximal Graphs](image)

**Maximal**

![Maximal Graphs](image)
\( U(n) \) = maximum size of a UDG of order \( n \).

The hypercube provides a lower bound on the number of unit distances \( \Omega(n \log n) \).

By considering points in a square grid with carefully chosen spacing, Erdős found an improved lower bound and offered a prize of \$500 for determining whether or not \( U(G) \) also has this form.

Erdős (1946): \( U(n) > n^{1+c/\log\log n} \).

Spencer, Szemeredi and Trotter (1984): \( U(n) < cn^{4/3} \).
Hypercube $Q_k$

$n = 2^k$

$e = k2^{k-1}$

$U(G) \geq n \log n / 2$
A UDG is maximal if the addition of any edge results in a graph which is not a UDG.

\[ u(n) = \text{minimum size of a maximal UDG of order } n. \]

Purdy and Purdy (1988): (CRAY computer)
Chilakamarri and Mahoney (1995): (by hand)
All maximal UDGs on at most 7 vertices.

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Some Maximal UDGs on 7 Vertices

$B_{7,12,1}$

$B_{7,11,1}$

$B_{7,11,2}$

$B_{7,11,3}$

$B_{7,11,4}$

$B_{7,11,5}$

$B_{7,11,6}$
The Minimal Forbidden Graphs on \( \leq 7 \) Vertices

- \( K_4 \)
- \( K_{2,3} \)
- \( F_{6,9,1} \)
- \( F_{7,10,1} \)
- \( F_{7,11,1} \)
- \( F_{7,11,2} \)
Separable UDGs

If $G$ is not connected, then $G$ is not a maximal UDG.

There are no separable UDGs with $< 8$ vertices.

There are 3 separable maximal UDGs on 8 vertices.
Main Goal
Construct all maximal UDGs of order 8.

Method:
1. Determine $u(8)$ and $U(8)$.
2. Generate all 2-connected graphs of order 8 and size $k \in [u(8), U(8)]$.
3. Test for minimal forbidden subgraphs.
4. Apply numerical methods to find drawing.
Formulation #1

Minimize \[
\sum_{ij \in E} \left[ 1 - \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \right]^2
\]
Formulation #1

Minimize \[ \sum_{ij \in E} [1 - \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}]^2 \]

Avoid non-UDG embeddings.
Formulation #2
Nonlinear Programming

Minimize \( \sum_{ij \in E} (b - z_{ij}) \)

\( z_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2, \quad \forall i, j \exists i < j \)

\( z_{ij} \geq 1, \quad \forall i, j \exists i < j \)

\( z_{ij} \leq b, \quad \forall ij \in E \exists i < j \)
61 Maximal UDGs of Order 8
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Rectilinear Drawing

Steinitz (1934) & Wagner (1936)
\[ cn(G) = 0 \implies rcn(G) = 0. \]

Bienstock & Dean (1993)
1. \[ cn(G) \leq 3 \implies cn(G) = rcn(G) \]
2. For every \( k \geq 4 \), there is a class of graphs \( G \) with \( cn(G) = k \) and \( rcn(G) \) unbounded.
The Complete Graph

cn(K₇) = rcn(K₇) = 9 \ (Guy \ -1971?)
rcn(K₁₀) = 62
(Brodsky, Durocher, Gethner \ -2000)

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Need Algebraic Characterization of Crossing

Crossing

No crossing
**Signed Area of a Triangle**

The signed area of a triangle can be calculated using the following formula:

\[
A2(i, j, k) = \begin{vmatrix}
1 & x_i & y_i \\
1 & x_j & y_j \\
1 & x_k & y_k \\
\end{vmatrix} = \begin{cases}
x_i y_j - x_j y_i \\
-x_i y_k + x_k y_i \\
x_j y_k - x_k y_j
\end{cases}
\]
Open Problems

• Given $G$ and an assignment for the binary variables, is there a feasible assignment for the real variables? For $G = K_n$?

• Let $D$ be a rectilinear drawing of $K_n$ with fewest crossings. Then the convex hull is a polygon. **Conjecture:** The convex hull is a triangle.
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Square of an Oriented Graph

- Square $G^2$ of a digraph $G = (V,E)$ is the digraph $(V, E \cup T)$ where $T = \{uv : d(u,v) = 2\}$.
- Seymour’s 2nd Neighborhood Conjecture: Every oriented graph has a node whose outdegree is at least doubled in its square.
- Dean’s conjecture: Every tournament has a node whose outdegree is at least doubled in its square.
Vertex of Minimum Degree

The center node $x$ has outdegree 5.

All others have outdegree $\geq 6$.

However, $d^{+2}(x) = 9$.

Conjecture: Every oriented graph has an independent set whose outdegree is at least doubled in its square.
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Cycle Modularity

• For positive integers $q, k$, a cycle of length $p$ is called a $(q \mod k)$-cycle if $p \equiv q \mod k$.

• **Open**: Characterize the 3-connected graphs which are pancyclic modulo 3.

• **Conjecture**: Every $k$-connected ($k \geq 3$) graph contains a $(0 \mod k)$-cycle.
  – Chen & Saito: $k=3$
  – Dean, Lesniak & Saito: $k=4$
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Chords of a Cycle

• Conjecture: Let $G$ be a $k$-connected ($k \geq 2$), non-hamiltonian graph. Then some cycle contains $k$ independent vertices and their neighbors.

• Cases $k = 2, 3$:
  Fournier (1982) and Manoossakis (1985)

• Thomassen’s Conjecture (1976): Every longest cycle in a 3-connected graph has a chord.

• Thomassen (1996): Every longest cycle in a 3-connected cubic graph has a chord.