Conjectures Resolved and Unresolved from Rigidity Theory

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Conjecture of Euler (1862): A plate and hinge framework that forms a closed polyhedron in 3-space is rigid.

Cauchy (1905): Any plate and hinge framework that forms a closed *convex* polyhedron in 3-space is rigid.

Gluk (1974): Almost all plate and hinge frameworks that form closed polyhedrons in 3-space are rigid.

Bob Connelly (1978): Constructs a counter-example to Euler's conjecture and exhibits it at an AMS conference at Syracuse University!

You can find the pattern to make your own model online: search for flexible polyhedra.
If the plates are all triangles, we may delete the plates and consider the underlying **rod and joint framework**.

**Rod and joint frameworks** in the plane and in 3-space have been studied extensively:

Start with a graph $\Gamma = (V, E)$.

Embed the vertex set in $\mathbb{R}^2$ or $\mathbb{R}^3$ and then “embed” the edges as **rigid rods** – the vertices represent **flexible joints**. (Rods are permitted to cross in the plane.)

Here are three “planar embeddings” of the same graph:
The rigidity or non-rigidity of the resulting framework is the same for all **generic** embeddings:

None of these are rigid

![None of these are rigid](image)

All of these are rigid

![All of these are rigid](image)

**Generic & Not rigid**  **Not Generic & Rigid**

![Generic & Not rigid](image) ![Not Generic & Rigid](image)

Generic rigidity is a **graph-theoretic concept**: A graph $\Gamma$ is **2-rigid** (**3-rigid**) if the generic embeddings of $\Gamma$ in $\mathbb{R}^2$ ($\mathbb{R}^3$) are rigid.
There should be combinatorial characterizations of 2-rigid and 3-rigid graphs.

Embedding the $n$ vertices of a graph in the plane, gives a framework with $2n$ degrees of freedom. As we include each rod, we expect to reduce the degrees of freedom by one:

Every rigid planar framework (except a single vertex) has 3 degrees of freedom. Hence, we expect that a graph on $n$ vertices will need $2n-3$ edges to be rigid. In fact, it is easy to prove that 2-rigid graph on $n$ vertices has at least $2n-3$ edges. A 2-rigid graph on $n$ vertices with exactly $2n-3$ edges is said to be minimally 2-rigid.
One might conjecture that $2n-3$ edges would be necessary and sufficient. However:

In the graph on the right, the second diagonal to the quadrilateral is **wasted**!

**Laman’s Theorem.** A graph $\Gamma = (V, E)$ on $n$ vertices with $2n-3$ edges is 2-rigid if and only if for every subset $U \subseteq V$ with $|U|>1$, we have $|E(U)| \leq 2|U|-3$.

In 3-space, each point has 3 degrees of freedom and a rigid body has 6 degrees of freedom. Hence it is natural to

**Conjecture:** A graph $\Gamma = (V, E)$ on $n$ vertices with $3n-6$ edges is 3-rigid if and only if for every subset $U \subseteq V$ with $|U|>2$, we have $|E(U)| \leq 3|U|-6$. 

2x5-3=7 and is rigid. But is not!
Unfortunately, there is a simple counterexample: the “double banana”

\[ |V| = 8 \]
\[ |E| = 18 \]
\[ = 3 \times 8 - 6 \]

and the Laman condition holds for all subsets of vertices.

Evidently there are other ways to waste edges in 3-space. But no one has been able to describe them and reformulate the Laman Conjecture for 3-space.
Another approach due to Henneberg: We can build a minimally 2-rigid graph from a single edge by a sequence of 2-attachments.

Clearly any graph constructed in this way will be minimally 2-rigid.
But, we can’t construct them all this way: the last vertex added always has degree 2 and

We need to be able to attach a vertex of degree 3. Given a minimally 2-rigid graph and 3 vertices $u$, $v$ and $w$, with the edge $uv$ included, deleting the edge $uv$ and attaching a new vertex $x$ to $u$, $v$ and $w$ is called a 3-attachment.
It is not too hard to show that any graph constructed from a single edge by a sequence of 2- and 3-attachments will be 2-rigid:

Since the average vertex degree of minimally 2-rigid graph is $2(2n-3)/n = 4 - (6/n)$, we can prove:

**Theorem.** A graph is minimally 2-rigid if and only if it can be constructed from a single edge by a sequence of 2- and 3-attachments.
The 3-dimensional analog to 2- and 3-attachments is 3- and 4-attachments:

And we can prove:

**Theorem.** A graph that can be constructed from a triangle by a sequence of 3- and 4-attachments will be minimally 3-rigid.

However, the average degree of a vertex of a minimally 3-rigid graph is $2(3n-6)/n$ or slightly less than 6. Hence to construct all minimally 3-rigid graphs we would have to be able to make 3-, 4- and 5-attachments.
Let $\Gamma$ be a minimally 3-rigid graph; a set of 5 vertices including two designated vertex disjoint edges is said to be available. A 5-attachment consists of attaching a degree 5 vertex to an available set of vertices, deleting the two designated edges.

It is not hard to prove:

**Theorem.** Every minimally 3-rigid graph can be constructed by a sequence of 3-, 4- and 5-attachments.
Conjecture (Henneberg). Every graph that can be constructed by a sequence of 3-, 4- and 5-attachments is minimally 3-rigid.

Note: the “double banana” could not be constructed by a sequence of 3-, 4- and 5-attachments. But perhaps some other pathological configuration could be constructed this way.

In general, one can consider making \((m+k)\)-extensions to minimally \(m\)-rigid graphs for any \(m\). In \(\mathbb{R}^m\) the number of rods in a minimally \(m\)-rigid framework is \(mn - (m+1)m/2\).
Note: 1-rigidity is simply connectivity and the minimally 1-rigid graphs are the trees.

An \((m+k)\)-extension to minimally \(m\)-rigid graph consists of deleting \(k\) appropriately chosen edges and attaching a new vertex of degree \(m+k\).

For example, to make a \((1+k)\)-extension to a tree, delete any \(k\) edges and attach a new vertex by an edge to each of the resulting \(1+k\) components. The result is clearly another tree.

Any graph constructed by a sequence of \((m+k)\)-extensions starting with \(K_m\) will have the correct number of edges to be minimally \(m\)-rigid and will satisfy the “Laman Condition” no subgraph has too many edges.

Any graph constructed by a sequence of \(m\)- and \((m+1)\)-extensions starting with \(K_m\) will be minimally \(m\)-rigid.
\((m+k)\)-extensions for \(k>2\) always result in minimally \(m\)-rigid graphs for \(m=1,2\) but can result in non-minimally \(m\)-rigid graphs for all \(m>2\). In particular, the double banana can be constructed from the triangle by a sequence of three 3-extensions followed by a 4-extension and then a 6-extension:
Will an \((m+k)\)-extension of an \(m\)-rigid graph always be \(m\)-rigid?

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*The extensions needed to construct all \(m\)-rigid graphs are highlighted in red.

The only case for which we do not know whether or not an \((m+k)\)-extension of an \(m\)-rigid graph will always be \(m\)-rigid is that of 5-extensions in 3-space!

**Millennium Problem #10**

Build a stronger mathematical theory for isometric and rigid embedding that can give insight into protein folding.