Are Almost All Graphs Determined by their Spectrum?

Chris Godsil

16 January, 2014
Outline

1 Against
   - Graphs and Matrices
     - Many Problems

2 For
   - Walk Matrices
   - Wang & Xu
Two graphs, two matrices

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\quad \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]
Isomorphic implies cospectral

If graphs $X$ and $Y$ are isomorphic, there is a permutation matrix $P$ such that

$$A(Y) = PA(X)P^T$$

Since permutation matrices are orthogonal, this means that $A(X)$ and $A(Y)$ are similar.

**Definition**

Graphs $X$ and $Y$ are *cospectral* if the characteristic polynomials of their adjacency matrices are equal.
Willem Haemers proposes:

**Conjecture**

The proportion of graphs on \( n \) vertices that are determined by their spectrum goes to 1 as \( n \to \infty \).
The Origin of our Difficulty

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There’s a problem...

\[ \phi(X, t) = t^5 - 4t^3. \]
If $m, n \geq 1$ then the graphs

$$K_{1,mn}, \quad K_{m,n} \cup (m - 1)(n - 1)K_1$$

are cospectral.
Did we forget to say connected?
Did we forget to say connected?
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<thead>
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We have forgotten complements, but...
The two graphs are cospectral, and so are their complements.
Latin square graphs: regularity is no help

16 squares as vertices; adjacent if in same row, same column, or same color. Cospectral.
Latin square graphs: regularity is no help

16 squares as vertices; adjacent if in same row, same column, or same color. Cospectral. (And there are lots of Latin squares.)
If we denote this tree by $S$, then $S \setminus u$ and $S \setminus v$ are cospectral. We say they are cospectral vertices in $S$. 
Schwenk’s Pairs of Cospectral Trees

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The bad news for trees

Theorem (Schwenk)

The proportion of trees on $n$ vertices that are determined by their spectrum goes to 0 as $n \to \infty$. 
Well, what does the computer tell us?

<table>
<thead>
<tr>
<th>$n$</th>
<th># graphs</th>
<th>$A$</th>
<th>$A$ &amp; $\overline{A}$</th>
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<tbody>
<tr>
<td>5</td>
<td>34</td>
<td>0.059</td>
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<td>6</td>
<td>156</td>
<td>0.064</td>
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<td>0.201</td>
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Source: Godsil and McKay, Haemers and Spence
Local Switching: regular and all, half, none

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We’ve seen it before

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Features of local switching

- Graphs related by local switching are cospectral, and their complements are too.
- The proportion of graphs on $n$ vertices that we can usefully apply local switching to goes to 0 as $n \to \infty$. 
Asymptotically useless, but good in the short run

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And if we look at graphs on 11 and 12 vertices...

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<td>12</td>
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Walk matrices

**Definition**

If $X$ is a graph on $n$ vertices and $\mathbf{1}$ is the all-ones vector, the walk matrix $\mathcal{W}$ of $X$ is the matrix with columns $A^r \mathbf{1}$ for $r = 0, \ldots, n - 1$.

**Definition**

A graph is **controllable** if its walk matrix is invertible.
My Conjecture

Conjecture

The proportion of graphs on $n$ vertices that are controllable goes to 1 as $n \to \infty$. 

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Theorem

Suppose $X$ and $Y$ are cospectral graphs with cospectral complements. If $X$ is controllable there is a unique orthogonal matrix $Q$ such that $Q^T A(X) Q = A(Y)$. The matrix $Q$ is rational and $Q1 = 1$.

Theorem

Suppose $X$ is controllable. If $Q$ is an orthogonal matrix such that $Q1 = 1$ and $Q^T A Q$ is an integer matrix, then $Q$ is rational.

Further, if $\ell \mathcal{W}^{-1}$ is an integer matrix, so is $\ell Q$. 
Our level best

Definition

The level of a rational matrix $M$ is the least integer $\ell$ such that $\ell M$ is an integer matrix.

Wang and Xu provide considerable evidence that if $X$ is controllable and $Q^T A(X) Q$ is an integer matrix then the level of $Q$ is two for a positive fraction of the graphs on $n$ vertices. They prove that orthogonal matrices of index two are essentially local switchings and that asymptotically these are negligible.

This provides evidence that a positive fraction of the graphs on $n$ vertices are determined by their spectrum.
We know very little about what happens when we use the Laplacian in place of the adjacency matrix.

What proportion of trees on $n$ vertices contain a pair of cospectral vertices?
The End(s)