generalized colorings of graphs

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A constructive theory of partitions, arranged in three acts, an interact and an exodion
A Constructive theory of Partitions, arranged in three Acts, an Interact and an Exodion.

By J. J. Sylvester, with Insertions by Dr. F. Franklin.

Act I. On Partitions Regarded as Entities.

... seeming parted,
But yet a union in partition.

Twelfth-night.

(1) In the new method of partitions it is essential to consider a partition as a definite thing, which end is attained by regularization of the succession of its parts according to some prescribed law. The simplest law for the purpose is that the arrangement of the parts shall be according to their order of magnitude. A leading idea of the method is that of correspondence between different complete systems of partitions regularized in the manner aforesaid. The perception of the correspondence is in many cases greatly facilitated by means of a graphical method of representation, which also serves per se as an instrument of transformation.

(2) The most obvious mode of graphically representing a partition is by means of a network or web formed by two systems of parallel lines or filaments. Any continuous portion of such web will serve to represent a partition, as ex gr. the graph

```
*   *   *
*   *   *
*   *   *
```

will represent the partition 3 5 5 4 3 of 20 by reading off the successive numbers of nodes parallel to the horizontal lines of the web. This, however, is not a regularized partition; the partition will be represented in its regularized form by such a graph as the following:

```
*   *   *
*   *   *
*   *   *
```
A traditional coloring
The *chromatic number* of G is the fewest number of colors needed to color the vertices of G so that each edge has vertices of two distinct colors on its ends.
The *chromatic number* of $G$ is the fewest number of colors needed to color the vertices of $G$ so that each edge has vertices of two distinct colors on its ends.
\( \chi(G) = 3 \)
The Hadwiger-Nelson Problem
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The Problem of Coloring the Plane
Problem 1. Let $G$ be the (infinite) graph where vertices are all points in the plane and vertices are adjacent if they are a distance of exactly one apart.

What is the chromatic number of $G$?
The Moser Spindle
John Isbell
Remark. The chromatic number of the plane is between four and seven
If each color class is Lebesque measurable then at least five colors are required.
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If each color class is surrounded by a Jordan Curve then at least six colors are required.
A finite plane graph
Theorem (de Bruin, Erdos, Furstenberg)

The problem is equivalent to finding the largest chromatic number of a finite plane graph.
Theorem (Coulson, Radoičić, Tóth)

The chromatic number of 3-space is between six and fifteen.
Problem 2
Tetrahedron
Glue convex polygons together.
Glue convex polygons together so
Glue convex polygons together so

- adjacent faces share a common edge

and

- are not coplanar
Glue convex polygons together to form an orientable surface.
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If it has genus $g$, call it an $S_g$-polytope.
An $S_0$-polytope
Fix $g$ and consider all $Sg$-polytopes.
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Consider the maximum chromatic number of all duals of all $Sg$-polytopes.
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Consider the maximum chromatic number of all duals of all Sg-polytopes.

Call this $\chi(Sg)$. 
\(\chi(S_0) = 4.\)
What is $\chi(S_g)$?
Theorem (Thomassen, jg)

\[ \chi(S_g) \leq c \ g^{3/5} \]
Question (Thomassen, jg).

Is \( \chi(S_g) \) bounded for all \( g \)?
Question (Thomassen, jg).

Is $\chi(S_g) \leq 10000$?
Problem 3
A *cocoloring* of $G$ is a partition of the vertices where each part induces a complete or empty graph.
The fewest number of parts needed is the *cochromatic number*, \( z(G) \), of \( G \).
$Z(G) \leq \chi(G)$
Let $R_n$ be the random graph on $n$ labeled vertices with edge probability one half.
Fact:

Almost surely

$$\chi \left( R_n \right) / Z(R_n) \rightarrow 1$$
Question (Erdös, jg):

Is it true that almost surely

\[ \chi (R_n) - Z(R_n) \to \infty \]
Prize:

Yes   $100

No    $1000
Problem 4
Problems 4
Let $\chi_k(G)$ be the fewest number of colors needed so that each color class induces a graph with maximum degree at most $k$. 
\[ \chi_2 = 2 \]
Let \( c_{\leq k}(G) \) be the fewest number of colors needed so that each monochromatic component has order \( k \) or less.
\chi(G) = \chi_0(G) = c_1(G)
\chi_{k-1}(G) \leq c_k(G)
Remark. For each surface $S$ there exists an $m$ so that if $G$ embeds on $S$ then $c_m(G) \leq 6$. 
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Questions (Chappell, jg) Can 6 be replaced with 5?
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Questions (Chappell, jg) Can 6 be replaced with 5?

Can 6 be replaced with 4?
Remark. If G is planar with girth 11 then $c_2(G) \leq 2$. 
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Question (Chappell, jg) Can 11 be reduced?
Theorem (Chappell, jg).

For fixed $k$, the problem $c_k(G) \leq 3$ is NP-complete, even when restricted to planar graphs.
Question. What girth restrictions will turn $c_k(G) \leq 2$ to P on planar graphs?
thank you
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