Some Things I Don’t Know

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slides available on DBW preprint page
The Pancake Problem [1975]

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Sorting by prefix reversal: Let permutations of \([n]\) be adjacent if they differ by reversing a prefix. What is the diameter \(f(n)\) of the resulting “pancake network”?
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Ex. 31452
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Ex.  31452 → 54132
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**Ex.** 31452 → 54132 → 23145
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Trivial: $n \leq f(n) \leq 2n - c$. 
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**Thm.** (Gates–Papadimitriou [1979]; Györi–Turán [1978])

$\frac{17}{16} n - c \leq f(n) \leq \frac{5}{3} n + c.$
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Thm. (Heydari–Sudborough [1997]) \(\frac{15}{14} n - c \leq f(n)\).
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**Thm.** (Chitturi–Fahle–Meng–Morales–Shields–Sudborough–Voit[2009]) \(f(n) \leq \frac{18}{11}n + c \approx 1.636n\).
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Conj. \(f(n) \sim \frac{3}{2} n\)
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- Eriksson–Eriksson-Karlander–Svensson–Wástlund [2001] \(\leq \left\lfloor \frac{2}{3}n - \frac{2}{3} \right\rfloor\) for sorting by block transpositions, via longer proof.
Number of \((r + 1)\)-cliques [1982]

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**Ques.** How many \((r + 1)\)-cliques must occur?
A Structural Variation

**Conj.** (West [1982]) If $G$ has $n$ vertices, maxdeg $D$, not $r$-majorizable, then $k_{r+1}(G) \geq (n - D)^t$, 
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**Sharp:** $G_{n,r,D}-z$ is $r$-partite: $t+1$ parts of size $n-D$, then strict increasing. All $(r+1)$-cliques use $z$, which neighbors all in the first $t$ parts and one in the others.

$$G_{19,5,16}$$

$$18 = 3 \cdot 5 + \binom{5-2}{2}$$
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\[
G_{19,5,16} = 3 \cdot 5 + \binom{5-2}{2} = 18
\]

**True:** for $r = 2$, for $t = 0$, and for $(r, n, D) = (3, 7, 5)$.
**Def.** (Bernhart–Kainen [1979]) book embedding: Order the vertices along the spine of a book, embed edges on pages. Each edge is on one page; edges on a page do not cross. \textit{pagenumber} = min \#pages.
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**Ex.** \( p(K_n) = \lfloor n/2 \rfloor \).

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\begin{center}
\begin{tikzpicture}
    
    \node[circle, draw, inner sep=2pt] (A) at (0,0) {};
    \node[circle, draw, inner sep=2pt] (B) at (1,0) {};
    \node[circle, draw, inner sep=2pt] (C) at (2,0) {};
    \node[circle, draw, inner sep=2pt] (D) at (3,0) {};
    \node[circle, draw, inner sep=2pt] (E) at (4,0) {};
    \node[circle, draw, inner sep=2pt] (F) at (5,0) {};
    
    \draw (A) -- (B);
    \draw (B) -- (C);
    \draw (C) -- (D);
    \draw (D) -- (E);
    \draw (E) -- (F);
\end{tikzpicture}
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**Pagename** [1988]

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**Ques.** (Leighton) What is \( p(K_n \square K_n) \)?
Acyclic Orientations [1995]

**Def.** An edge in an acyclic orientation is dependent if reversing it creates a cycle. Let $d_{\min}(G)$ and $d_{\max}(G)$ be the min & max #dependent edges in orientations of $G$. 
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**Def.** $G$ is fully orientable if $\exists$ acyclic orientation with $k$ dependent edges whenever $d_{\text{min}}(G) \leq k \leq d_{\text{max}}(G)$. 
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**Def.** $G$ is **fully orientable** if $\exists$ acyclic orientation with $k$ dependent edges whenever $d_{\text{min}}(G) \leq k \leq d_{\text{max}}(G)$.

**Ques.** Which graphs are fully orientable? Bipartite?
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**No:** Turán graph $T_{n,r}$ when $r \mid n$ (Chang–Lin–Tong [’09]).
**Def.** Let $l(n, k)$ be the largest $t$ such that every connected $n$-vertex graph with minimum degree at least $k$ has a spanning tree with at least $t$ leaves (and hence connected domination number $\leq n - t$).
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```
  \[ \frac{k-2}{k+1} n + 2 \]
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- $l(n, k) \geq \frac{k-2}{k+1} n + c$ for $k \leq 4$ (Kleitman–West [1991]) and $k \in \{4, 5\}$ (Griggs–Wu [1992]).
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\[ \begin{array}{c}
 k+1 \quad k+1 \quad k+1 \quad k+1 \\
 \end{array} \]

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**Thm.** (Caro–West–Yuster [2000]) \( l(n, k) \sim n \frac{k-\ln(k+1)}{k+1} \).
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**Thm.** (Caro–West–Yuster [2000]) $l(n, k) \sim n^{\frac{k-\ln(k+1)}{k+1}}$.

**Ques.** How does $\frac{l(n,k)}{n}$ decline from $\frac{k-2}{k+1}$ to $\frac{k-\ln(k+1)}{k+1}$?
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\[
\begin{align*}
G & \quad p(G) \leq 4 \\
\quad & \quad \text{not spec} \\
\quad & \quad \hat{p}(G) = 5 \\
P_{18} & \quad p(P_{18}) = 5
\end{align*}
\]
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**Ex.** \( p(P_n) = \lceil \lg n \rceil \).

**Conj.** \( \hat{p}(G) = p(G) \) for every bipartite \( G \).
$p(G)$ when $G$ is dense

**Ex.** Give the vertices of $K_{2^k}$ distinct $k$-tuple binary codes. Color $E(K_{2^k})$ by giving $uv$ the color $u \oplus v$.

\[
\begin{array}{c c c}
01 & & 11 \\
00 & & 10 \\
\end{array}
\]

\[
\begin{array}{c c}
\text{-purple} & = 01 \\
\text{blue} & = 11 \\
\text{red} & = 10 \\
\end{array}
\]
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<table>
<thead>
<tr>
<th>01</th>
<th>11</th>
</tr>
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<tbody>
<tr>
<td>00</td>
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</tr>
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\[ = 01 \]
\[ = 11 \]
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**Thm.** (Bunde–Milans–Wu–West [2008]) \( \hat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1 \).
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01 & 10 & 00 & 11 \\
\end{array}
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![Diagram showing vertex coloring]

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**Conj.** $p(K_n) = \hat{p}(K_n) = 2^\lceil \lg n \rceil - 1$. True for $n \leq 16$. 
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00 & 10 & 11 \\
\end{array}\]

\[00 \quad 01 \quad 10 \quad 11\]

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A more detailed conjecture for \( \hat{p}(K_{r,s}) \) would strengthen "Yuzvinsky’s Theorem" on sums of subsets of \( \mathbb{F}_2^k \).
The Reconstruction Problem

**Def.** The deck of a graph $G$ is the multiset of cards of the form $G - v$ for $v \in V(G)$. 
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![Graphs with different decks](image)

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**Obs.** $|E(G)| = \frac{\sum_v |E(G-v)|}{n-2}$ when $G$ has $n$ vertices.

This info is lost when keeping only some cards.
Degree-Associated Reconstruction [2010]

**Def.** (Ramachandran [1981]) the dacards are the pairs $(G - v, d_G(v))$ for $v \in V(G)$. The degree-associated reconstruction number $drn(G)$ is the minimum number of dacards in a multiset that determines $G$. 
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** Conj. ** $drn(G) ≤ \frac{n}{4} + 2$ when $G$ has $n$ vertices.
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**Ques.** Must equality hold when \(G\) has no “twins”?
More on $\text{drn}(G)$

- $\text{drn}(tK_m) = 3$ (Ramachandran [2006]) but $\text{rn}(tK_m) = m + 2$ (Myrvold [1989]).
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**Ques.** What other graphs satisfy $\text{rn}(G) - \text{drn}(G) > 1$?
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**Thm.** (Myrvold [1990]) If $T$ is a tree with at least five vertices, then $\rn(T) = 3$. 
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```
    H_1
       /
      /  \
     /    \
H_2
```

**Thm.** For caterpillars, $\text{drn}(T) = 2$ unless $T$ is a star or $H_1$. 
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![Diagram of trees](attachment:diagram.png)

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\[
\begin{align*}
\text{H}_1 & \quad \text{H}_2
\end{align*}
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- Hannah Spinoza has extended the upper bound to “subdivided caterpillars with toes”.

\[
\begin{align*}
\text{H}_1 & \quad \text{H}_2
\end{align*}
\]
Nine Dragon Tree Conjecture [2010]

Aim: Common generalization of Nash-Williams’ Formula and decomposition results for planar graphs.
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**Aim:** Common generalization of Nash-Williams’ Formula and decomposition results for planar graphs.

**Thm.** (Nash-Williams [1965]) $G$ decomposes into $k$ forests $\iff |E(H)| \leq k(|V(H)|−1)$ for every subgraph $H$. 
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**Def.** Fractional arboricity $\text{Arb}(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)| - 1}$.

(Payan [1986])
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(Payan [1986]) N-W: arboricity $\gamma(G) = \lceil \text{Arb}(G) \rceil$. 
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(Payan [1986]) \( \text{N-W}: \) arboricity \( \Upsilon(G) = \lceil \text{Arb}(G) \rceil \).

**Idea:** Three forests are needed when \( \text{Arb}(G) = 2 + \varepsilon \); can we restrict the third forest?
Nine Dragon Tree Conjecture [2010]

**Aim:** Common generalization of Nash-Williams’ Formula and decomposition results for planar graphs.

**Thm.** (Nash-Williams [1965]) $G$ decomposes into $k$ forests $⇔ |E(H)| ≤ k(|V(H)|−1)$ for every subgraph $H$.

**Def.** fractional arboricity $\text{Arb}(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|−1}$.

(Payan [1986]) N-W: arboricity $\gamma(G) = \lceil \text{Arb}(G) \rceil$.

**Idea:** Three forests are needed when $\text{Arb}(G) = 2 + \varepsilon$; can we restrict the third forest?

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**Nine Dragon Tree (NDT) Conjecture:**
(Montassier, Ossona de Mendez, Raspaud, Zhu [2010]) $\text{Arb}(G) \leq k + \frac{d}{k+d+1} \Rightarrow G$ decomposes into $k+1$ forests, with the last being $d$-bounded.