Directionally Sensitive Multivariate Statistical Process Control Methods

Ronald D. Fricker, Jr.
Naval Postgraduate School
October 5, 2005

Abstract

In this paper we develop two directionally sensitive statistical process control procedures by modifying Hotelling’s $\chi^2$ procedure and a multivariate CUSUM procedure by Crosier. We then compare the performance of these procedures via simulation to the original directionally invariant procedures and to simultaneous univariate directionally sensitive Shewhart and CUSUM procedures. The results show, not unexpectedly, that the modified multivariate procedures work better than their original counterparts in the problem for which they were designed. Interestingly, the results of comparing simultaneous univariate Shewhart procedures to the modified $\chi^2$ was mixed, with the better procedure depending upon the covariance structure of the distribution. In contrast, the modified MCUSUM generally outperformed the simultaneous univariate CUSUMs for all covariance structures we considered. Furthermore, the modified MCUSUM also performed better than the univariate Shewharts and the modified $\chi^2$. These results thus suggest that the modified MCUSUM procedure is the preferred choice (from among the procedures considered here) for monitoring multivariate processes in a particular direction.

1 Introduction

Many existing multivariate statistical process control (SPC) methods are directionally invariant, meaning they are designed to detect changes in a mean vector in all directions equally well. Examples of such procedures include Hotelling’s $\chi^2$ [3], Crosier’s Multivariate CUSUM [1], and more recently the nonparametric method of Qui and Hawkins [8]. See Lowry and Montgomery [5] for a more detailed discussion. The lack of directional sensitivity is often seen as a limitation of these methods, particularly when practitioners are interested in detecting changes in some directions more than others.

For example, the Centers for Disease Control and Prevention (CDC) as well as many state and local health departments around the United States have started to develop and field syndromic surveillance systems. Making use of existing health care or other data, often already in electronic form, these surveillance systems are intended to give early warnings of bioterrorist attacks or other emerging health conditions. (See Stoto, Fricker, et al. [10] for a more detailed discussion.)

With such syndromic surveillance systems, it is important to quickly flag increases in the relevant measures because, in terms of signaling a terrorist event, decreases are generally irrelevant. Current syndromic surveillance systems tend to run multiple simultaneous univariate schemes, each focused on detecting an increase in a single dimension.

Multiple simultaneous univariate schemes have the advantages of ease of implementation and interpretation, but they can be less sensitive to some types of changes when compared to multivariate methods. Further, unless the signal thresholds of the multiple simultaneous procedures are properly set, they can suffer from a higher than desired combined false alarm rate.

In this paper, we present modifications to two multivariate methods – Hotelling’s $\chi^2$ and a multivariate CUSUM (MCUSUM) by Crosier [1] – to make them directionally sensitive and then illustrate their performance via simulation. The modifications are motivated by the procedures’ univariate counterparts and how those counterparts achieve directionality.

- The univariate counterpart to Hotelling’s $\chi^2$ is the Shewhart procedure [9] where directionality is achieved by signaling only when an observation falls far enough out in one particular tail of the distribution.
For the “modified Hotelling’s χ²,” directionality is achieved by only signaling when an observation falls within a particular region of space corresponding to a tail region of the multivariate distribution.

- The CUSUM is, of course, the univariate counterpart to the MCUSUM, where directionality (and signaling speed) is achieved by reflecting the CUSUM at zero in either the positive or negative direction combined with an appropriate signal threshold in that direction. For the “modified MCUSUM,” directionality is achieved by reflecting each component of the cumulative sum vector at zero in the desired direction combined with an appropriate signal threshold.

In this paper we first describe the two standard univariate procedures (Shewhart and CUSUM) followed by their multivariate counterparts (Hotelling’s χ² and Crosier’s multivariate CUSUM). We then describe how to modify the multivariate procedures to make them directionally sensitive and compare and contrast the various procedures’ performance via simulation.

2 Notation and Terminology

In the simple case of detecting a shift from one specific distribution to another, let F₀ denote the in-control distribution, which is the desired or preferred state of the system. For syndromic surveillance, for example, this could be the distribution of the daily counts of individuals diagnosed with a particular complaint at a specific hospital or within a particular geographic region under normal conditions. Let F₁ denote the out-of-control distribution where, under the standard SPC paradigm, this would be a particular distribution representing a condition or state that is important to detect. Within the syndromic surveillance problem, F₁ might represent an elevated mean daily count resulting from the release of a bioterrorism pathogen for example.

Let ν be the actual (unknown) time when the process shifts from F₀ to F₁ and let T be the length of time from ν to when a procedure signals (referred to as the delay). The notation Eν(T|T ≥ 0) is used to indicate the expected delay, which is the average time it takes an procedure to signal once the shift has occurred. The notation E∞(T) indicates the expected time to a false alarm, where ν = ∞ means the process never shifts to the out-of-control distribution.

In the SPC literature, procedures are compared in terms of the expected time to signal, where E∞(T) is first set equally for two procedures and then the procedure with the smallest Eν(T|T ≥ 0), for a particular F₁, is deemed better. Often when conducting simulation comparisons, ν is set to be 0, so the conditioning in the expectation is automatic.

The term average run length (ARL) is frequently used for the expected time to signal, where it is understood that when ν = ∞ the ARL denotes the expected time to false alarm. In simulation experiments, the performance of various procedures is compared by setting the expected time to false alarms to be equal and then comparing ARLs when ν = 0, where it is then understood that the ARL is the mean delay time.

3 Univariate Procedures

3.1 Shewhart’s Procedure

Shewhart’s procedure [9] is probably the simplest and best known of all SPC methods. The basic idea is to sequentially evaluate one observation (or statistic) at a time, signaling when an observation that is rare under F₀ occurs. The most common form of the procedure, often known as the X chart, signals when the absolute value of an observed sample mean exceeds a pre-specified threshold c, often defined as the mean value plus some number of standard deviations of the mean.

More sophisticated versions of the Shewhart procedure exist that look for increases in variation and other types of out-of-control conditions. These versions are not considered here in order to keep the evaluations simple and consistent with Hotelling’s χ² procedure.

The Shewhart procedure can be made directionally sensitive by only signaling for deviations in one direction. For example, in syndromic surveillance, only deviations in the positive direction that would indicate a potential outbreak are assumed to be important to
detect. Thus, for a univariate random variable \( X \), and for some desired probability \( p \), the threshold \( c \) is chosen to satisfy
\[
\int_{x=c}^{\infty} f_0(x) \, dx = p.
\]
The algorithm proceeds by observing values of \( X_i \); it stops and concludes \( X_i \sim F_1 \) at the first time \( i \) when \( X_i > c \).

If the change to be detected is a one-time jump in the mean and the probability of an observation exceeding the threshold is known, then simulation is not required as the delay is geometrically distributed and exact calculations for the average run lengths can be directly calculated as \( \mathbb{E}_\infty(T) = 1/p \) and
\[
\mathbb{E}_\nu(T|T \geq 0) = \mathbb{E}_0(T) = \left[ \int_{x>c}^{\infty} f_1(x) \, dx \right]^{-1}.
\]

### 3.2 CUSUM Procedure

The CUSUM of Page [6] and Lorden [4] is a sequential hypothesis test for a change from a known in-control density \( f_0 \) to a known alternative density \( f_1 \). The procedure monitors the statistic \( S_i \), which satisfies the recursion
\[
S_i = \max(0, S_{i-1} + L_i),
\]
where the increment \( L_i \) is the log likelihood ratio
\[
L_i = \log \left( \frac{f_1(X_i)}{f_0(X_i)} \right).
\]
The procedure stops and concludes that \( X_i \sim F_1 \) at the first time \( i \) for which \( S_i > c \), where \( c \) is some prespecified threshold that achieves a desired ARL under the in-control distribution.

If \( F_0 \) and \( F_1 \) are normal distributions with means \( \mu \) and \( \mu + \delta \), respectively, and unit variances, then (1) reduces to
\[
S_i = \max(0, S_{i-1} + (X_i - \mu) - k),
\]
where \( k = \delta/2 \). This is the form commonly used, even when the underlying data is only approximately normally distributed.

Note that, since the univariate CUSUM is “reflected” at zero, it is only capable of looking for departures in one direction. If it is necessary to guard against both positive and negative changes in the mean, then one must simultaneously run two CUSUMs, one of the form in (2) to look for changes in the positive direction, and one of the form
\[
S_i = \max(0, S_{i-1} - (X_i + \mu) - k),
\]
to look for changes in the negative direction. When directional sensitivity is desired, say to detect only positive shifts in the mean, it is only necessary to use (2).

### 4 Directionally Invariant Multivariate Procedures

#### 4.1 Hotelling’s \( \chi^2 \)

Hotelling [3] introduced the \( \chi^2 \) procedure (often referred to as the \( T^2 \) procedure; we use \( \chi^2 \) to indicate that we are assuming the covariance matrix is known). For multivariate observations \( X_i \in \mathbb{R}^d \), \( i = 1, 2, \ldots \), the procedure computes
\[
\chi_i^2 = X_i' \Sigma^{-1} X_i,
\]
where \( \Sigma^{-1} \) is the inverse of the covariance matrix. The procedure stops at the first time \( i \) for which \( \chi_i > c \), where \( c \) is a pre-specified threshold.

Like the original univariate Shewhart \( \bar{X} \) procedure, because it only uses the most recent observation to decide when to stop, the \( \chi^2 \) can react quickly to large departures from the in-control distribution but will also be relatively insensitive to small shifts. Of course, it also requires that the covariance matrix is known or well estimated.

#### 4.2 Crosier’s MCUSUM

The abbreviation MCUSUM, for multivariate CUSUM, is used here to refer to the procedure proposed by Crosier [1] that at each time \( i \) considers the statistic
\[
S_i = (S_{i-1} + X_i - \mu)(1-k/C_i), \quad \text{if } C_i > k,
\]
where \( k \) is a statistical distance based on a predetermined vector \( k \), \( k = (k' \Sigma^{-1} k)^{1/2} \) and \( C_i = (S_{i-1} + X_i - \mu)' \Sigma^{-1} (S_{i-1} + X_i - \mu) \) is determined. If \( C_i \leq k \) then reset \( S_i = 0 \). The procedure starts with \( S_0 = 0 \) and sequentially calculates
\[
Y_i = (S_i' \Sigma^{-1} S_i)^{1/2}.
\]
It concludes that \( X_i \sim F_1 \) at the first time \( i \) when \( Y_i > c \) for some threshold \( c > 0 \).
Crosier proposed a number of other multivariate CUSUM-like algorithms but generally preferred (3) after extensive simulation comparisons. Pigatiello and Runger [7] proposed other multivariate CUSUM-like algorithms as well, but found that they performed similar to (3).

It is worth noting that Crosier derived his procedure in an ad hoc manner, not from theory, but found it to work well in simulation comparisons. Healy [2] derived a sequential likelihood ratio test to detect a shift in a mean vector of a multivariate normal distribution that is a true multivariate CUSUM. However, while Healy’s procedure is more effective (has shorter ARLs) when the shift is to the pre-decrease, it is less effective than Crosier’s for detecting other types of shifts, including mean shifts that were close to but not precisely the specific mean vector of F1.

5 Directionally Sensitive Multivariate Procedures

5.1 Modified Hotelling’s $\chi^2$

To modify Hotelling’s $\chi^2$ procedure to achieve directional sensitivity, we modify the stopping rule so that it meets two conditions: (1) $\chi_i > c$ and (2) $X_i \in S$, where $S$ is a particular subspace of $\mathbb{R}^d$ that corresponds to, say, a positive shift in one or more components of the mean vector. In syndromic surveillance, this would correspond to an increase in one or more disease indicators, for example.

For the purposes of the simulations to follow, $S$ was defined as follows. Choose values $s_1, s_2, \ldots, s_d$ such that

$$\int_{x_1=s_1}^{\infty} \int_{x_2=s_2}^{\infty} \cdots \int_{x_d=s_d}^{\infty} f_0(X) dx \approx 0.99$$

and then define $S = \{x_1 > s_1, x_2 > s_2, \ldots, x_d > s_d\}$.

For example, consider an in-control distribution following a bivariate normal distribution with some positive correlation, so that the probability contours for the density of $F_0$ is an ellipse with its main axis along 45-degree line in the plane. Then you can think about $S$ as the upper the upper right quadrant that almost encompasses the 99 percent probability ellipse.

The idea of using this region for $S$ is that if $F_1$ represents an increase in one or more components of the $F_0$ mean vector, then the modified $\chi^2$ procedure will have an increased probability of signaling, which should result in a decreased expected time to signal. On the other hand, if $F_1$ represents a condition with a mean vector that corresponds to a decrease in one or more of the $F_0$ mean vector components, then the probability of signaling will decrease and the procedure will have less of a chance of producing a signal.

5.2 Modified MCUSUM

Unlike other multivariate CUSUMs (e.g., Healy’s [2]), Crosier’s MCUSUM formulation is easy to modify to only look for positive increases. In particular, for detecting positive increases, such as in the syndromic surveillance problem, when $C_i > k$ we limit $S_i$ to be positive in each dimension by replacing (3) with $S_i = \{s_{i,1}, \ldots, s_{i,d}\}$ where

$$S_{i,j} = \max[0, (S_{i-1,j} + X_{i,j} - \mu_j)(1 - k/C_i)],$$

for $j = 1, 2, \ldots, d$.

The motivation for this modification follows directly from the univariate CUSUM’s reflection at 0. As in (2), the reflection helps ensure that a large cumulative sum vector causing a signal is the result of a mean shift in one or more positive directions.

6 Results: Performance Comparisons via Simulation

These simulations compare the performance by average run length, first setting the ARL under the in-control distribution (i.e., $\mathbb{E}_0(T)$, the expected time to false alarm) equally, and then comparing the ARL performance under numerous out-of-control distributions resulting from various shifts in the mean vector at time 0 (i.e., $\mathbb{E}_{F_0}(T)$).

The in-control distribution ($F_0$) is a six-dimensional multivariate normal with a zero mean vector, $\mu_0 = \{0, 0, 0, 0, 0, 0\}$ and a covariance matrix $\Sigma$ consisting of unit variances on the diagonal and constant covariance $\rho$ on the off-diagonals. The out-of-control distributions ($F_1$s) have the same covariance structure.
but with the mean vector shifted by some distance \(d\),

\[
|\vec{\mu}_0 - \vec{\mu}_1| = \left(\sum_{i=1}^{n} (\mu_1(i))^2\right)^{1/2} = d,
\]

where the shift occurs in some number of dimensions \(n\), \(1 < n \leq 6\). For those dimensions with a shift, the shifts were made equally:

\[
\mu_1(1) = \cdots = \mu_1(n) = \sqrt{\frac{d^2}{\pi}}.
\]

The simulations were conducted in Mathematica where the random observations were generated using the \textit{MultinormalDistribution} function. The in-control ARLs were set to 100 by empirically determining the threshold for each procedure via simulation. For the multivariate procedures this involved determining a single threshold for each value of \(\rho\) (except for Hotelling’s \(\chi^2\) procedure for which one threshold applies to all values of \(\rho\)).

For the simultaneous univariate procedures approach, which requires a separate threshold for each individual procedure, there was no reason to favor one direction over another, so all the thresholds were set such that the probability of false alarm was equal in all dimensions and so that the resulting expected time to false alarm for the combined set of univariate procedures was equal to the expected time to false alarm of the multivariate procedure.

In addition, for the univariate CUSUMs we set \(k = 0.5\) in (2). For the MCUSUM and modified MCUSUM we set \(k = \{0.2, 0.2, 0.2, 0.2, 0.2, 0.2\}\) in (3).

Generally, it is quite simple to empirically estimate the ARLs via simulation. For a particular \(F_0\), choose a \(c\) and run the particular procedure \(m\) times, recording for each run the time \(t\) when the first \(X_i > c\) (where each \(X_i\) is a random draw from \(F_0\), of course). Estimate the in-control ARL as

\[
\overline{E}_{\infty}(T) = \frac{\Sigma t}{m}
\]

adjusting \(c\) and re-running as necessary to achieve the desired in-control ARL, where \(m\) is made large enough to make the standard error of \(\overline{E}_{\infty}(T)\) acceptably small.

Having established the threshold \(c\) for that \(F_0\) with sufficient precision, then for each \(F_1\) of interest re-run the algorithm \(n\) times (where \(n\) is often smaller than \(m\)), drawing the \(X_i\)s from \(F_1\) starting at time 1. As before, take the average of \(ts\) to estimate the expected delay.

In the simulations to follow, we first demonstrate the modified procedures’ performance compared to their counterpart unmodified procedures. This establishes the directional sensitivity and effectiveness of the modified procedures. We follow this with comparisons of the modified procedures to procedures consisting of the application of simultaneous univariate procedures. The simultaneous univariate procedures are implemented to be directionally sensitive in the same direction as the modified multivariate procedures and hence provide some indication about the effectiveness of the modified multivariate methods. Finally, we compare the best procedures from the previous comparisons in an effort to determine whether a single procedure is generally best.

### 6.1 Modified Procedures vs. Original Procedures

Figure 1 shows the improved performance of the modified \(\chi^2\) procedure and the modified MCUSUM for almost all types of mean vector shifts where, as previously described, the componentwise shifts are in the positive direction. This is not surprising given that the modified procedures were designed to look for positive mean shifts.

In Figure 1, the various lines correspond to the number of dimensions in \(\vec{\mu}_1\) that shifted and the horizontal axis is the distance of the mean shift. For example, the \(n = 1\) line shows the results for \(\vec{\mu}_1 = \{d, 0, 0, 0, 0, 0\}\), where the ARL was evaluated at \(d = 0.0, 0.2, 0.4, \ldots, 3.4\). Similarly, the \(n = 2\) line shows the results for \(\vec{\mu}_1 = \{\sqrt{\frac{d}{2}}, \sqrt{\frac{d}{2}}, 0, 0, 0, 0\}\).

The vertical axis is the difference \(\Delta\) between the ARL for the original procedure and the modified procedure for a given mean vector shift. Positive values indicate the modified procedure had a smaller ARL, so that for a particular out-of-control condition the modified procedure had a shorter time to signal. A \(\Delta = 0\) at \(d = 0.0\) indicates that the false alarm rates (equivalently, the in-control ARLs) were set equally for each procedure before comparing the expected time to signal for various \(\vec{\mu}_1\)s (within the bounds of experimental error, where a sufficient number of simulation runs were conducted to achieve a standard error of \(\Delta\) of approximately 2 percent of the estimated in-control ARLs.)
Figure 1: Performance comparison of the $\chi^2$ and MCUSUM procedures vs. their modified counterpart procedures for $\rho = 0.3$. A positive value of $\Delta$ indicates that the ARL for the modified procedure is shorter than the ARL of its unmodified counterpart.

Figure 2: Performance comparison of the modified $\chi^2$ procedure vs. multiple simultaneous univariate Shewhart procedures. The figure on the left shows that for $\rho = 0.3$ the multiple simultaneous Shewhart procedures give smaller ARLs for all $n$. However, the figure on the right with $n = 3$ shows that either procedure can be significantly better than the other depending on the value of $\rho$. 
Figure 3: Performance comparison of the modified MCUSUM procedure vs. multiple simultaneous univariate CUSUM procedures. The figure on the left shows that for $\rho = 0.3$ the MCUSUM does better for small values of $d$ and marginally worse for large $d$. However, unlike the modified $\chi^2$ in Figure 2, the figure on the right for the modified MCUSUM with $n = 3$ is generally better than simultaneous univariate CUSUMs for all values of $\rho$.

Figure 4: Performance comparison of the modified MCUSUM procedure to multiple simultaneous univariate Shewhart procedures (left graph) for $\rho = 0.3$ and to the modified $\chi^2$ procedure (right graph) for $\rho = 0.9$. In both cases, the modified MCUSUM procedure performs generally better than the preferred Shewhart-type procedure.
As previously mentioned, the expected result that the modified procedures generally outperform the original procedures at detecting positive shifts. Figure 1 shows this for the case of $\rho = 0.3$. Not shown, the results for other values of $\rho$ we tried, from $\rho = 0$ to $\rho = 0.9$, are very similar.

In particular, the modified $\chi^2$ outperforms Hotelling's $\chi^2$ for all combinations of $1 < n \leq 6$, $0.0 < d \leq 3.4$, and $0 \leq \rho \leq 0.9$. The modified MCUSUM outperforms Crosier's MCUSUM except for larger values of $\rho$ with small $d$ and small $n$.

For example, in Figure 1, Crosier's MCUSUM slightly outperforms the modified MCUSUM for $n = 1$ with $0 < d < 0.6$ or so. For $\rho = 0.6$, Crosier's MCUSUM outperforms the modified MCUSUM with $-6 < \Delta < 0$ or so for $n = 1,2$ with $0 < d < 0.6$. And, for $\rho = 0.9$, Crosier's MCUSUM outperforms the modified MCUSUM with $-9 < \Delta < 0$ or so for $n = 1, \ldots, 5$ with $0 < d < 1$.

In this work, we are interested in moderate values of $\rho$, since the syndromic surveillance data we have seen has only had moderate correlations, roughly on the order of $0 < r < 0.6$. In addition, we are interested in shifts in the mean vector that exhibit themselves as small changes in multiple dimensions. (Indeed, if the shift is expected only a small number ($n$) of dimensions or that the covariance $\rho$ will be high, then it's likely that univariate methods would be more appropriate anyway.)

With this in mind, what is most notable in Figure 1 is that as $n$ increases the modified procedures do considerably better than their counterparts, particularly for moderate $d$s.

### 6.2 Modified Procedures vs. Univariate Procedures

Given that the modified $\chi^2$ performs better than the original $\chi^2$ for this problem, Figure 2 focuses on comparing the performance of the modified $\chi^2$ to six one-sided Shewhart procedures operating simultaneously. The left-side graph of Figure 2, constructed just like Figure 1, shows that six simultaneous univariate Shewharts are more effective (have shorter ARLs) than the modified $\chi^2$ for $\rho = 0.3$. At best, for large shifts, the ARL of the modified $\chi^2$ approaches the performance of the multiple univariate Shewharts, and for small to moderate shifts the multiple univariate Shewharts are clearly better.

The graph on the right side of Figure 2 shows the performance comparison for $n = 3$ and for various values of $\rho$ ($0.0$, $0.3$, $0.6$, and $0.9$). Here we see that the better procedure depends on $\rho$, where the modified $\chi^2$ is better for values of $\rho$ near $0.0$ or $0.9$ while the simultaneous univariate Shewharts are better for moderate values of $\rho$. Most notably, and a bit surprisingly, the modified $\chi^2$ significantly outperforms the simultaneous univariate Shewharts when there is no correlation ($\rho = 0$) and when the shift is only in $3$ of the $6$ dimensions.

The results for the modified MCUSUM versus simultaneous univariate CUSUMs are presented in Figure 3. These results differ from those for the Shewhart-type procedures in Figure 2 in that the modified MCUSUM is generally better than the simultaneous univariate CUSUMs regardless of the value of $\rho$. In particular, in the left graph of Figure 3 the modified MCUSUM performance when $\rho = 0.3$ is somewhat better for small shifts (roughly $> 0.0$ to $0.6$ or so), and slightly worse than multiple univariate CUSUMs for moderate to large shifts. Yet, in the figure at the right we see that the modified MCUSUM is better for small shifts for all values of $\rho$ and only performs slightly worse for moderate values of $\rho$ combined with moderate to large values of $d$. 
6.3 Modified MCUSUM vs. Best Other Procedures

What the previous simulations have shown is that the modified MCUSUM is generally better than the simultaneous univariate CUSUMs. However, whether the modified $\chi^2$ is better than simultaneous univariate Shewharts depends on $\rho$. So, here we compare the modified MCUSUM to both the simultaneous univariate Shewharts when it is better than the modified $\chi^2$ ($\rho = 0.3$) and the modified $\chi^2$ when it is better then the simultaneous univariate Shewharts ($\rho = 0.9$). The results are shown in Figure 4. In both comparisons, the modified MCUSUM procedures performance is better. The obvious conclusion, then, is a preference for the modified MCUSUM that, at least in these simulations for a jump change in the mean vector of multivariate normal distributions.

Now, all the figures up to this point have displayed differences in ARL performance between two procedures. Figure 5 shows the ARLs for the modified MCUSUM for $n = 3$. Results for $n = 1, 2, 4, 5, 6$ were similar, though the individual $\rho$ curves moved around but largely stayed within the same band/region. For example, for $n = 1$, the lowest ARLs were achieved with $\rho = 0.9$ while for $n = 6$ the lowest ARLs were achieved with $\rho = 0.0$.

7 Discussion and Conclusions

In this paper we have demonstrated how to modify two directionally invariant multivariate procedures to make them directionally sensitive. The results of these and other simulations not included here show, not unexpectedly, that the modified multivariate procedures work better than their original counterparts in the problem for which they were designed. It is not unexpected since the modified procedures specifically look for positive changes so that, when given such changes, should outperform their counterparts that are not so designed.

In the comparison of simultaneous univariate Shewhart procedures to the modified $\chi^2$ the outcomes were mixed, with the better procedure depending upon the distribution covariance structure (i.e., $\rho$). In contrast, the modified MCUSUM generally performed better than the simultaneous univariate CUSUMs for all values of $\rho$ that were considered. Furthermore, it also performed better than the Shewharts and the modified $\chi^2$ except in those cases where the shift $d$ was moderate to very large (in which case a statistical detection procedure may not even be required). These results thus suggest that, among the procedures considered here, the modified MCUSUM procedure is the preferred choice for monitoring multivariate processes for positive shifts in the mean.

It is important to recognize that the use of the modified MCUSUM does come with some costs. First, unlike Hotelling’s $\chi^2$ procedure, the choice of threshold, and hence the ARL performance of the procedure, depends on the covariance structure of the data. This limitation is also true of the multiple simultaneous univariate procedures, though if practitioners treat each dimension separately it is less apparent. Second, practitioners are often less comfortable using multivariate procedures because they tend to feel such procedures do not provide sufficient information about the cause(s) of a signal. The modified MCUSUM is no different in this regard, though because it is directional, the practitioner is at least assured that the signal is related to an out-of-control condition of interest.

7.1 Directions for Future Research

Future work should consider the effects of estimation on the performance of the procedures. In particular, the multivariate procedures require the estimation of the entire covariance matrix while the simultaneous univariate procedures only require estimation of the diagonal elements. Whether and how this estimation affects the performance of the procedures is not known. In addition, this work assumed a non-negative, constant $\rho$ covariance structure. However, it is conceivable that some problems might involve both positive and negative covariances of varying magnitudes. In addition, while the effects of changing $k$ in the CUSUM are well known, the effects of changes in $k$ in the MCUSUM are not as well known and those effects were not explored in this work. Finally, the method we applied to make the multivariate procedures directionally sensitive can be
applied to other directionally invariant procedures, such as the nonparametric method of Qui and Hawkins [8]. How the performance of those new methods compares to the performance of the modified MCUSUM requires further research.

References


