1. Signals

1.2 Elementary Signals

Whenever we try to attach numerical quantities to a phenomenon, we need a reference frame. This starts from the moment we learn how to count, when our fingers become our own reference frame, to the time we learn how to write numbers when the powers of ten become the reference frame. The worse which can happen to a person is to find him/herself into a situation where there is no reference frame: in the middle of the desert or of the ocean without instruments, or among people speaking a totally foreign language. In other words without a reference frame we are lost!

In signal processing we have the same problem: we need an appropriate reference frame so that we can place the signals we try to analyze. Since we deal with signals, any reference frame of interest has to be made of signals with well know properties, which reflect the kind of information we want to extract. For example if we hear a musical note, and we want to determine which note it is in the scale, we need to compare it with sinusoidal signals (ie "tones") at all frequencies and determine which frequency is the closest.

In what follows we present some of the elementary discrete time signals which will be at the basis of the analysis developed in the rest of the book.

1.2.1 Discrete Time Unit Impulse

The unit impulse $\delta[n]$ is the most "elementary" signal, and it provides the simplest expansion. It is defined as

$$
\delta[n] = \begin{cases} 
1 & \text{if } n = 0 \\
0 & \text{otherwise}
\end{cases}
$$

and it is plotted in figure 1.2.1, together with a general shifted version $\delta[n-k]$.
Any discrete time signal can be expanded into the superposition of elementary shifted impulses, each one representing each of the samples. This is expressed as
\[ x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n-k] \]
where each term \( x[k] \delta[n-k] \) in the summation expresses the \( n \)-th sample of the sequence.

**Example:** the sequence shown in figure 1.2.2 can be expanded as
\[ x[n] = 1.5 \delta[n+2] - 1.0 \delta[n+1] + 1.2 \delta[n] - 0.5 \delta[n-1] + 0.5 \delta[n-2] + 1.6 \delta[n-3] \]
This expansion can be viewed as the following animation (double click the figure to animate)
The significance of this expansion is the fact that any signal, no matter how complicated, is decomposed into elementary pulses properly scaled in amplitude and shifted in time.

Of particular interest is the Unit Step sequence

$$u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

shown below. It is a simple exercise to show that

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$
### 1.2.2 Continuous Time Unit Impulse: the Dirac Function

In continuous time we have to be a bit more careful. A unit impulse $\delta(t)$ is defined as a function which is zero for all $t \neq 0$, and yet its integral is nonzero. In particular $\delta(t)$ is such that

$$\delta(t) = 0 \quad \text{for all } t \neq 0,$$

and it is represented as in figure 1.2.3. It can be viewed as the limit of a sequence of a rectangular signal of width $T$ and height $\frac{1}{T}$, as $T \to 0$.

![Figure 1.2.3: Dirac "Delta" Function](image)

Its significance is the fact that for any signal $x(t)$, continuous at time $t$, we can write

$$\int_{-\infty}^{+\infty} x(t - \tau) \delta(\tau) \, d\tau = \int_{0^-}^{0^+} x(t - \tau) \delta(\tau) \, d\tau = x(t) \int_{0^-}^{0^+} \delta(\tau) \, d\tau = x(t)$$

This is known as the "Sifting Property":

$$x(t) = \int_{-\infty}^{+\infty} x(t - \tau) \delta(\tau) \, d\tau = \int_{-\infty}^{+\infty} x(\tau) \delta(t - \tau) \, d\tau$$

The rightmost integral in the above expression is easily derived by a change in the integration variable.

Comparing to the equivalent expression in discrete time, which we recall here for convenience,

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n - k]$$

we can see that in both cases we expand a signal (continuous time or discrete time) in terms of a sequence of unit impulses.

Of particular interest is the Unit Step signal defined as

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

shown below. It is a simple exercise to show that
As we will be seeing in the Fourier Analysis section, sinusoidal signals are one of the building blocks of more general signals. For example an audio signal is made of vibrations, of the vocal cords in humans and animals, or strings or standing waves for musical instruments. When we stand next to a car playing rock music at a full blast we feel the vibrations right in our own stomach!

A continuous time sinusoidal signal is defined as

\[ x(t) = A \cos(\omega_0 t + \alpha) \]

with \( A \) being the Amplitude. If the independent variable \( t \) denotes time, and it is measured in seconds, then \( \omega_0 \) indicates the angular frequency in radians per second, and \( \alpha \) the phase in radians. Since the cosinus, as all trigonometric functions, does not change when the argument is shifted by multiples of \( 2\pi \), we can see the periodicity of the sinusoidal signal from the expression

\[ x(t) = A \cos(\omega_0 t + \alpha + 2\pi) = A \cos(\omega_0 (t + T_0) + \alpha) = x(t + T_0) \]

where \( T_0 = \frac{2\pi}{\omega_0} \) is the period in seconds (if \( t \) is in seconds). Amplitude and period are shown in figure 1.2.4 below.
A discrete time sinusoid is obtained by sampling a continuous time sinusoid with sampling interval $T_s$, as

$$x[n] = x(nT_s) = A \cos(\Omega_0 T_s n + \alpha)$$
By defining the digital frequency \( \omega_0 = \Omega_0 \frac{T_s}{T_s} \), we can write the sampled sinusoid as

\[
x[n] = A \cos (\omega_0 n + \alpha)
\]

As we can see from the definition, the digital frequency \( \omega_0 = \Omega_0 \frac{T_s}{T_s} = 2\pi F_0 \frac{T_s}{T_s} = \frac{\text{rad}}{\text{sec}} \times \text{sec} \) has no dimensions. In order to see the meaning of the digital frequency, define the sampling frequency \( F_s = \frac{1}{T_s} \), which represents the number of samples taken every second, and write it as

\[
\omega_0 = 2 \pi \frac{F_0}{F_s} \text{ radians}
\]

In other words, \( \omega_0 \) is a relative frequency, with respect to the sampling frequency.

Example: a sinusoid with frequency \( F_0 = 2 \text{ kHz} \) is sampled every \( T_s = 0.1 \text{ msec} = 10^{-4} \text{ sec} \). Then the sampling frequency is \( F_s = 10^4 \text{ Hz} = 10 \text{ kHz} \), and the digital frequency of the sinusoid is \( \omega_0 = 2 \pi \frac{F_0}{F_s} = 2 \pi \frac{2000}{10000} = \frac{2\pi}{5} \text{ radians} \)

Homework Problems: Problems 1.3, 1.4

See if you are following. Click HERE for some Questions (not graded)
1.2.4 Complex Exponentials

Although we represent signals in terms of sinusoidal signals (read vibrations), trigonometric functions do not lend themselves to easy manipulations. Fortunately, a sinusoid can be expanded in terms of complex exponentials, which, on the other hand, exhibit very convenient mathematical properties. In particular recall Euler’s formula (or one of the many Euler’s formulas):

\[
\cos(\alpha) = \frac{1}{2} (e^{j\alpha} + e^{-j\alpha}) \\
\sin(\alpha) = \frac{1}{2j} (e^{j\alpha} - e^{-j\alpha})
\]

for any angle \( \alpha \). Therefore, substituting for the appropriate time varying angle, both continuous time and discrete time sinusoidal signals can be expressed in terms of complex exponential signals as in the following:

\[
x(t) = A\cos(2\pi F_0 t + \alpha) = \frac{A}{2} e^{j\alpha} e^{j2\pi F_0 t} + \frac{A}{2} e^{-j\alpha} e^{-j2\pi F_0 t}
\]

\[
x[n] = A\cos(\omega_0 n + \alpha) = \frac{A}{2} e^{j\alpha} e^{j\omega_0 n} + \frac{A}{2} e^{-j\alpha} e^{-j\omega_0 n}
\]

The reason why the exponential signal in general is more attractive than the sinusoidal signal, is because a number of significant operations we perform on signals become just algebraic manipulations in complex exponentials. For example:

a) **Differentiation and Integration.** Take any signal \( x(t) \) and compute its derivative or its integral with respect to time, as \( y_d(t) = \frac{d}{dt} x(t) \) or \( y_I(t) = \int x(t) dt \). In general both \( y_d(t) \) and \( y_I(t) \) have expressions different from \( x(t) \), UNLESS \( x(t) \) is an exponential, in which case differentiation and integration are just algebraic operations as multiplication and division:

\[
y_d(t) = \frac{d}{dt} x(t) = \frac{d}{dt} \left( e^{j2\pi F_0 t} \right) = \left( j2\pi F_0 \right) e^{j2\pi F_0 t} = (\frac{j2\pi F_0}{j2\pi F_0}) x(t)
\]

\[
y_I(t) = \int x(t) dt = \int e^{j2\pi F_0 t} dt = \left( \frac{1}{j2\pi F_0} \right) e^{j2\pi F_0 t} = (\frac{1}{\frac{1}{j2\pi F_0}}) x(t)
\]

b) **Time Shift.** Similarly, take any sequence \( x[n] \) and shift it in time as \( y[n] = x[n - L] \), with \( L \) being an integer. Again if \( x[n] \) is an exponential signal (and ONLY in this case), then the shifted sequence is obtained just by multiplication, as

\[
y[n] = x[n - L] = e^{j\omega_0 (n-L)} = (e^{-j\omega_0 L}) e^{j\omega_0 n} = (e^{-j\omega_0 L}) x[n]
\]

In more technical words we say that the complex exponential signal is an *eigenfunction* of Differential, Integral and Time Shift operators. This means that, ONLY in the case of exponentials, these operations are just algebraic manipulations. All other signals are not so lucky! This is at the basis of most (not all) the transform techniques (Fourier, Laplace, \( z \)-, and all their relatives) which are introduced as tools to analyze signals, systems and their interactions.
1.2.5 Signal Manipulations

Given a continuous time or a discrete time signal $x(t)$ or $x[n]$, we define a number of operations, which we list here for convenience. Apart from obvious algebraic manipulations, like scaling by a constant, we have to be particularly careful when we manipulate the time axis. In particular we define:

a) **Time Shift**: $x(t-t_0)$ or $x[n-n_0]$, with $t_0$ (real) or $n_0$ (integer) representing a time shift. This is shown in figure 1.2.6;

![Figure 1.2.6: Shift in Time](image)

b) **Time Scaling**: this is easily defined in continuous time as $x(at)$. However in discrete time, since the independent variable has to be an index, in general it is not defined, apart from some particular cases which we are not going to present here. In the continuous time case, according to whether the constant $a$ is positive or negative, we have two different situations: in the latter ($a < 0$) the signal is also inverted on the time axis. This is shown in figure 1.2.7 below.
Figure 1.2.7: Scaling of Time Axis

Homework Problems