D2. Beamforming

Objectives

- Understand a simple mathematical model for propagation through a medium;
- Understand Demodulation using one receiver
- Understand Demodulation using multiple receivers
- Beamforming for separating signals from different directions

1. Introduction

   The previous chapter ended with the effects of modulation in the frequency domain. The main idea is that, by modulation, you can shift the frequency spectrum of a signal so that it can be transmitted through a medium. In this chapter we extend this concept by introducing a simple mathematical model of a propagating signal and its transmission by modulation and demodulation. It is shown that, with equally spaced multiple receivers, signals coming from different directions can be separated and processed independently. It is like having a directional antenna which is electronically steered by the proposed choice of combining coefficients.

2. Transmission of a Signal by Propagation

   **VIDEO: Propagation (10:02)**
   [http://faculty.nps.edu/rcristi/eo3404/d-beamforming/videos/d2_02_propagation.mp4](http://faculty.nps.edu/rcristi/eo3404/d-beamforming/videos/d2_02_propagation.mp4)

   Information is transmitted by propagating a signal through a medium, which can be air, water and so on. In the case of wireless propagation, the wavefront from a transmitting antenna, in the absence of local reflections, begins as a spherical pattern and, at a distance, it becomes close to a planar wavefront. This defines the “near field” (spherical waves, close to the transmitter) and the “far field” (planar waves, far from the transmitting antenna). This is shown in the figure below.
Spherical Waves in the Near Field and Planar Waves in the Far Field

If we call $x(t)$ and $y(t)$ the transmitted and received signals respectively, the simplest effect of the channel is given by the following relation

$$y(t) = Ax(t - T_d) + \eta(t)$$

where $A$ represents the “attenuation” due to the energy dispersed during propagation, $T_d$ is a time delay, due to the time it takes for the signal to propagate, and $\eta(t)$ is random noise from the medium and/or interferences. Specially in the underwater environment, a more accurate model also includes multipath, due to reflections from the bottom and the surface and also different paths due to temperature gradient of the water. This goes beyond the scope of this course, but the reader should be aware of this.

**VIDEO: Mod. and Dem.: One Receiver (17:06)**
http://faculty.nps.edu/rcristi/eo3404/d-beamforming/videos/d2_03_oneReceiver.mp4

A general scheme showing the modulator at the transmitter and the demodulator at the receiver is seen in the figure below.

**Modulator at the Transmitter and Demodulator at the Receiver**

Recall from the previous section, that the effect of the modulator is to shift the signal in the frequency domain, as
where $S(F) = FT\{s(t)\}$. The demodulator shifts the signal back to the original spectrum and the demodulated signal has the following form

$$r(t) = As(t - T_d)e^{j\phi}e^{-j2\pi f_c T_d}$$

where, for simplicity, we ignore the effect of the noise $\eta(t)$. This would be a situation with a strong received signal with high SNR. In this expression we see again the attenuation $A$ and the time delay $T_d$ due to the channel (as we saw before). Also we see that there are two time shifts: $e^{\pi df_c T_d}$, due to the effect of the time delay of the carrier in the propagation, and also an other term $e^{j\phi}$ which accounts for synchronization errors between the transmitter and the receiver.

The whole modulation process is not difficult to see, but it is beyond the scope of these notes and it is presented in the Appendix for the interested reader.

### 3. Beamforming

An interesting effect is obtained when we use a number of receivers, all equally spaced from each other. By doing this, we explore small time delays caused in the receivers by a waveform coming at an angle so that they can be combined constructively, to enhance the signal, or destructively, to eliminate it.

#### 3.1 Multiple Receivers

**VIDEO: Multiple Receivers (12:30)**  
http://faculty.nps.edu/rcristi/eo3404/d-beamforming/videos/d2_04_multipleReceivers.mp4

Let us see a general scheme of $N$ equally spaced receivers as shown in the figure below.
Planar waveform received by a number of receivers: the time delay changes with the angle of arrival

Looking at this figure and using simple geometry, we can see that there is a time delay $T$ between the signals at two adjacent receivers. For example

$$r_1(t) = r_0(t - T)$$

where

$$T = \frac{d \sin(\theta_0)}{c}$$

with $c$ being the velocity of propagation in the medium. Recursive application shows that the demodulated signal at the $k$-th receiver can be expressed as

$$r_k(t) = r_0(t - kT)$$

for $k = 0,...,N-1$. Since

$$r_0(t) = As(t - T_d)e^{j\phi}e^{-j2\pi f_c T_d}$$

then we can easily see that

$$r_k(t) = r_0(t - kT) = As(t - T_d - kT)e^{j\phi}e^{-j2\pi f_c (T_d + kT)} \
\approx As(t - T_d)e^{j\phi}e^{-j2\pi f_c T}$$

where we used the assumption

$$s(t - T_d - kT) \approx s(t - T_d)$$

since the time delays are small as

$$kT << 1/B$$

The above expression implies finally that

$$r_k(t) \approx r_0(t)e^{-j2\pi f_c T}, k = 0,...,N-1$$
This is our fundamental equation, which shows that, after demodulation, all the received signals are related by a phase shift. In order to give a better expression to this signal, let us define

\[ u_0 = 2\pi F_c T \]

then substitute for the time delay \( T \) to obtain

\[ u_0 = 2\pi F_c \frac{d \sin(\theta_0)}{c} \]

Since \( c / F_c = \lambda \), the wavelength of the carrier frequency, we obtain

\[ u_0 = 2\pi \frac{d}{\lambda} \sin(\theta_0) \]

and therefore

\[ r_k(t) \approx r_0(t)e^{-jku_0}, k = 0, ..., N - 1 \]

So, this is a key result to remember:

**Given**
- \( N \) equally spaced receivers (as shown in the figure above), spaced by a distance \( d \),
- with \( \lambda = c / F_c \) the wavelength of the carrier (\( c, F_c \) velocity of propagation and carrier frequency respectively);
- with planar wavefront angle of arrival \( \theta_0 \) as defined in the figure

then the demodulated signals \( r_0(t), r_1(t), ..., r_k(t), ..., r_{N-1}(t) \) are related as

\[ r_k(t) \approx r_0(t)e^{-jku_0}, k = 0, ..., N - 1 \]

\[ u_0 = 2\pi \frac{d}{\lambda} \sin(\theta_0) \]

In the next subsection we take advantage of this result to filter the incoming signals based on the direction of arrival.
### 3.2 Beamforming

**VIDEO: Beamforming (18:32)**

[http://faculty.nps.edu/rcristi/eo3404/d-beamforming/videos/d2_05_beamforming.mp4](http://faculty.nps.edu/rcristi/eo3404/d-beamforming/videos/d2_05_beamforming.mp4)

Now let us all linearly combine all received signals as in the figure below.

![General structure for Beamforming](image)

**General structure for Beamforming**

We linearly combine all the outputs of the receiver into one signal as

\[
y(t) = \sum_{k=0}^{N-1} w_k r_k(t)
\]

If there is a signal coming from direction \( \theta_0 \), then the received signal is

\[
y(t) = \sum_{k=0}^{N-1} w_k r_k(t) \approx W(u_0) r_0(t)
\]

where

\[
W(u) = \sum_{k=0}^{N-1} w_k e^{-jk\lambda}
\]

and (recall again)

\[
u_0 = \frac{2\pi d}{\lambda} \sin(\theta_0)
\]
Recall that $r_0(t)$ depends on the transmitted signal, as
\[ r_0(t) = Ae^{j\gamma} s(t - T_d) \]
where $\gamma$ includes all the phase shifts due to timing errors, delays and so on. Then we finally obtain
\[ y(t) \approx AW(u_0)e^{j\gamma} s(t - T_d) + \sum_{k=0}^{N-1} w_k \eta_k(t) \]
where we included also the noise $\eta_k(t)$ at the k-th receiver.

Notice that the function $W(u)$ has the same form as the DTFT of the sequence $...0,0,w_0,...,w_{N-1},0,0,...$. It is periodic with period $u = 2\pi$, since
\[ W(u) = W(u + 2\pi) \]
As a consequence, in order to have a one to one correspondence between the angle of arrival $\theta$ and $W(u)$, we need the normalized special frequency $u$ to be such that
\[ -\pi < u = 2\pi \frac{d}{\lambda} \sin(\theta) < +\pi \]
as $-\pi/2 < \theta < +\pi/2$. This implies that the spacing between adjacent receiving antennas has to be such that
\[ d \leq \lambda/2 \]

### 3.3 Choice of the weights
The function
\[ W(u) = \sum_{k=0}^{N-1} w_k e^{-jku} \]
shows the gain of the array at different angles of arrival $\theta$, with
\[ u = 2\pi \frac{d}{\lambda} \sin(\theta) \]
If we want to enhance a signal coming from an angle $\theta_0$ then we choose
\[ w_k = \frac{1}{N} e^{jku_0}, \text{ for } k = 0,...,N-1 \]
This yields, using the geometric series
\[ W(u) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j(k(N-u))} = \frac{1-e^{jN(u-u)}}{N(1-e^{j(N-u)})} \]

The magnitude of this expression is plotted (in dB’s) in the figure below, for two values of the number of receivers: \( N = 10, 50 \).

**Plot of the array gain as a function of** \( u_0 - u \)

For example, if we choose a 10 element array, with \( d = \lambda / 2 \) and \( \theta_0 = 30^\circ \) we obtain the gain pattern shown the figure below.

**Gain pattern for a steering direction** \( \theta_0 = 30^\circ \) **of a 10 element array.**
VIDEO: Matlab Example (18:50)
http://faculty.nps.edu/rcristi/eo3404/d-beamforming/videos/matlab2.mp4