Chapter 3: Problem Solutions

Fourier Analysis of Discrete Time Signals

Problems on the DTFT: Definitions and Basic Properties

■ Problem 3.1

Problem

Using the definition determine the DTFT of the following sequences. It it does not exist say why:

a) $x[n] = 0.5^n u[n]$

b) $x[n] = 0.5^{|n|}$

c) $x[n] = 2^n u[-n]$

d) $x[n] = 0.5^n u[-n]$

e) $x[n] = 2^{|n|}$

f) $x[n] = 3(0.8)^{|n|} \cos(0.1\pi n)$

Solution

a) Applying the geometric series

$$X(\omega) = \sum_{n=0}^{+\infty} 0.5^n e^{-j\omega n} = \frac{1}{1 - 0.5\, e^{-j\omega}}$$

b) Applying the geometric series

$$X(\omega) = \sum_{n=0}^{+\infty} 0.5^n e^{-j\omega n} + \sum_{n=-1}^{-\infty} 0.5^{-n} e^{-j\omega n} =$$

$$\sum_{n=0}^{+\infty} 0.5^n e^{-j\omega n} + \sum_{n=0}^{+\infty} 0.5^n e^{j\omega n} - 1 = \frac{1}{1-0.5\,e^{-j\omega}} + \frac{1}{1-0.5\,e^{j\omega}} - 1$$

$$= -\frac{1.5\,e^{i\,\omega}}{(-2.+e^{i\,\omega})\,(-0.5+e^{i\,\omega})}$$

c) Applying the geometric series

$$X(\omega) = \sum_{n=0}^{-\infty} 2^n e^{-j\omega n} = \sum_{n=0}^{\infty} 2^{-n} e^{j\omega n} = \frac{1}{1-0.5\,e^{j\omega}}$$

d) Applying the geometric series

$$X(\omega) = \sum_{n=0}^{-\infty} 0.5^n e^{-j\omega n} = \sum_{n=0}^{\infty} 0.5^{-n} e^{j\omega n} = \sum_{n=0}^{\infty} 2^n e^{j\omega n} \rightarrow \text{does not converge}$$

and the DTFT does not exist;

e) Applying the geometric series

$$X(\omega) = \sum_{n=0}^{+\infty} 2^n e^{-j\omega n} + \sum_{n=-1}^{-\infty} 2^{-n} e^{-j\omega n} = \sum_{n=0}^{\infty} 2^n e^{-j\omega n} + \sum_{n=1}^{+\infty} 2^n e^{j\omega n} \rightarrow \text{does not converge}$$

and the DTFT does not exist;

f) Expanding the cosine, we can write

$$x[n] = \frac{3}{2}\,0.8^n \times e^{j0.1\pi n} u[n] + \frac{3}{2}\,1.25^n \times e^{j0.1\pi n} u[-n-1] +$$
$$\frac{3}{2}\,0.8^n \times e^{-j0.1\pi n} u[n] + \frac{3}{2}\,1.25^n \times e^{-j0.1\pi n} u[-n-1]$$

which becomes

$$x[n] = \frac{3}{2}\,(0.8 \times e^{j0.1\pi})^n u[n] + \frac{3}{2}\,(1.25 \times e^{j0.1\pi})^n u[-n-1] +$$
$$\frac{3}{2}\,(0.8 \times e^{-j0.1\pi})^n u[n] + \frac{3}{2}\,(1.25 \times e^{-j0.1\pi})^n u[-n-1]$$

Taking the DTFT of every term, recall that

$$\text{DTFT}\{a^n u[n]\} = \sum_{n=0}^{+\infty} a^n e^{-j\omega n} = \frac{e^{j\omega}}{e^{j\omega}-a},\ \ \text{if}\ \ |a| < 1$$

$$\text{DTFT}\{a^n u[-n-1]\} = \sum_{n=-1}^{-\infty} a^n e^{-j\omega n} = \sum_{n=0}^{+\infty} a^{-n} e^{j\omega n} - 1 = -\frac{e^{j\omega}}{e^{j\omega}-a},\ \text{if}\ \ |a| > 1$$

Therefore we obtain

$$X(\omega) = \frac{e^{j\omega}}{e^{j\omega}-0.8\times e^{j0.1\pi}} + \frac{e^{j\omega}}{e^{j\omega}-0.8\times e^{-j0.1\pi}} - \frac{e^{j\omega}}{e^{j\omega}-1.25\times e^{j0.1\pi}} - \frac{e^{j\omega}}{e^{j\omega}-1.25\times e^{-j0.1\pi}}$$

■ Problem 3.2

Problem

Given the fact that DTFT $\{0.8^n u[n]\} = 1 / (1 - 0.8 e^{-j\omega})$ and using the properties, compute the DTFT of the following sequences:

a) $x[n] = 0.8^n u[n-2]$

b) $x[n] = 0.8^n u[n] \cos(0.1 \pi n)$

c) $x[n] = n0.8^n u[n]$

d) $x[n] = 0.8^{-n} u[-n]$

e) $x[n] = \begin{cases} 0.8^n & \text{if } 0 \le n \le 5 \\ 0 & \text{otherwise} \end{cases}$

Solution - Call $s[n] = 0.8^n u[n]$.

a) $x[n] = 0.8^n u[n-2] = 0.8^2 \; 0.8^{n-2} u[n-2]$

$$= 0.64 \; s[n-2]$$

Then $X(w) = 0.64 \, e^{-j2w} S(w) = 0.64 \, \dfrac{e^{-j2w}}{1 - 0.8 e^{-jw}}$ //

b) $x[n] = 0.8^n u[n] \cos(0.1\pi n)$

$$= \frac{1}{2} s[n] \, e^{j0.1\pi n} + \frac{1}{2} s[n] \, e^{-j0.1\pi n}$$

which implies $X(w) = \frac{1}{2} S(w - 0.1\pi) + \frac{1}{2} S(w + 0.1\pi)$

Substitute for $S(w)$ to obtain

$$X(w) = \frac{1}{2} \frac{1}{1 - 0.8 e^{-j(w - 0.1\pi)}} + \frac{1}{2} \frac{1}{1 - 0.8 e^{-j(w + 0.1\pi)}} \quad \text{//}$$

c) $x[m] = m \, 0.8^m \, u[m] = m \, s[m]$

From the properties of the DTFT:

$$X(w) = j \frac{d}{dw} S(w) = j \frac{d}{dw} \left(1 - 0.8 e^{-jw} \right)^{-1} =$$

$$= -j \left(1 - 0.8 e^{-jw} \right)^{-2} \left(j 0.8 \, e^{-jw} \right) \quad \longrightarrow$$

Therefore

$$X(w) = \frac{0.8\, e^{-jw}}{\left(1 - 0.8\, e^{-jw}\right)^2} \qquad //$$

d) $x[m] = 0.8^{-m}\, u[-m] = s[-m]$

Then

$$X(w) = \sum_{m=-\infty}^{+\infty} x[m]\, e^{-jwm} = \sum_{m=-\infty}^{+\infty} s[-m]\, e^{-jwm} = \sum_{m=-\infty}^{+\infty} s[m]\, e^{-j(-w)m} =$$

$$= S(-w) = \frac{1}{1 - 0.8\, e^{jw}} \qquad //$$

e) $\quad x[m] = \begin{cases} 0.8^m & 0 \le m \le 5 \\ 0 & \text{otherwise} \end{cases}$

we can write it as

$x[m] = 0.8^m \left(u[m] - u[m-6] \right)$

$\quad = 0.8^m u[m] - 0.8^m u[m-6]$

$\quad = 0.8^m u[m] - 0.8^6 \, 0.8^{m-6} u[m-6] =$

$\quad = s[m] - 0.8^6 \, s[m-6]$

Therefore

$$X(w) = \left(1 - 0.8^6 e^{-j6w}\right) S(w) =$$

$$= \frac{1 - 0.8^6 e^{-j6w}}{1 - 0.8 e^{-jw}} \quad //$$

Basic Problems on the DFT

■ Problem 3.3

Problem

Compute the DFT of the following sequences

a) $x = [1, 0, -1, 0]$

b) $x = [j, 0, j, 1]$

c) $x = [1, 1, 1, 1, 1, 1, 1, 1]$

d) $x[n] = \cos(0.25\,\pi n)$, $n = 0, \ldots, 7$

e) $x[n] = 0.9^n$, $n = 0, \ldots, 7$

Solution

a) $N = 4$ therefore $w_4 = e^{-j2\pi/4} = -j$. Therefore
$X[k] = \sum_{n=0}^{3} x[n]\,(-j)^{nk} = 1 - (-j)^{2k}$, $k = 0, 1, 2, 3$. This yields

$$X = [0, 2, 0, 2]$$

b) Similarly, $X[k] = j + j\,(-j)^{2k} + (-j)^{3k} = j + j\,(-1)^k + j^k$, $k = 0, \ldots, 3$, which yields

$$X = [1 + 2\,j, \ j, \ -1 + 2\,j, \ -j]$$

c) $N = 8$ and $w_8 = e^{-j2\pi/8} = e^{-j\pi/4}$. Therefore, applying the geometric sum we obtain

$$X[k] = \sum_{n=0}^{7} (w_8)^{nk} = \begin{cases} \frac{1-(w_8)^{8k}}{1-(w_8)^k} = 0 & \text{when } k \neq 0 \\ 8 & \text{when } k = 0 \end{cases}$$

since $(w_N)^N = (e^{-j2\pi/N})^N = 1$. Therefore

$$X = [8, 0, 0, 0, 0, 0, 0, 0]$$

d) Again $N = 8$ and $w_8 = e^{-j\pi/4}$. Therefore

$$X[k] \;=\; \tfrac{1}{2}\left(\sum_{n=0}^{7} e^{j\frac{\pi}{4}n}\, e^{-j\frac{\pi}{4}nk} + \sum_{n=0}^{7} e^{-j\frac{\pi}{4}n}\, e^{-j\frac{\pi}{4}nk}\right) \;=$$

$$=\; \tfrac{1}{2}\left(\sum_{n=0}^{7} e^{j\frac{\pi}{4}(k-1)n} + \sum_{n=0}^{7} e^{j\frac{\pi}{4}(-k-1)n}\right)$$

Applying the geometric sum we obtain

$$X = [0, 4, 0, 0, 0, 0, 4]$$

e) $X[k] = \sum\limits_{n=0}^{7} 0.9^{n}\, e^{-j\frac{\pi}{4}nk} = \dfrac{1-(0.9)^{8}}{1-0.9\, e^{-j\frac{\pi}{4}k}}$, for $k = 0, \ldots, 7$ Substituting numerically we obtain

$$X = [5.69,\ 0.38 - 0.67\, j,\ 0.31 - 0.28\, j,$$
$$0.30 - 0.11\, j,\ 0.30,\ 0.30 + 0.11\, j,\ 0.31 + 0.28\, j,\ 0.38 + 0.67\, j]$$

▪ Problem 3.4

Problem

Let X = [1, j, −1, −j], H = [0, 1, −1, 1] be the DFT's of two sequences x and h respectively. Using the properties of the DFT (do not compute the sequences) determine the DFT's of the following:

Solution:

a) $x[(m-1)_4]$ — Recall $DFT\{x[(m-L)_N]\} = e^{-jк\frac{2\pi}{N}L} X[к]$

In this case $N = 4$, $L = 1$ and

$DFT\{x[(m-1)_4]\} = e^{-jк\frac{2\pi}{4}} X[к] = (-j)^к X[к]$, $к = 0, .., 3$

This yields: $[1, 1, 1, 1]$ //

b) $\text{DFT}\{x[(m+3)_4]\} = e^{jK\frac{2\pi}{4}3}X[K] = j^{3K}X[K], \quad K = 0,...,3$

This yields the same answer as in a) above:

$$[1, 1, 1, 1].$$

In fact, notice that

$$x[(m-1)_4] = [x[3], x[0], x[1], x[2]] = x[(m+3)_4] \quad //$$

c) $Y[K] = H[K]X[K] = [0, j, 1, -j]$

d) $DFT\{(-1)^m x[m]\}$.

Recall that $DFT\{e^{jk_0 \frac{2\pi}{N} n} x[m]\} = X[(k-k_0)_N]$.

In this case, $N=4$ and therefore

$DFT\{(-1)^m x[m]\} = DFT\{e^{j2\frac{2\pi}{4} m} x[m]\} = X[(k-2)_4]$

which yields

$$[-1, -j, 1, j] \,/\!/$$

e) $\text{DFT}\{ j^m x[m]\} = \text{DFT}\{ e^{j\frac{2\pi}{4}m} x[m]\} =$

$$= X[(k-1)_4] = [-j, 1, j, -1] \;//$$

f) $\text{DFT}\{ x[(-m)_4]\}.$

By the definition of the DFT

$$\text{DFT}\{ x[(-m)_4]\} = \sum_{m=0}^{3} x[(-m)_4] e^{-j\frac{2\pi}{4}km} =$$

$$= x[0] + \sum_{m=1}^{3} x[4-m] e^{-j\frac{2\pi}{4}km} \quad . \text{ Now call } 4-m = m$$
$$\text{in the } \Sigma \longrightarrow$$

$$= x[0] + \sum_{m=1}^{3} x[m] \; e^{-j\frac{2\pi}{4}\kappa(4-m)} =$$

$$= x[0] + \sum_{m=1}^{3} x[m] \; e^{j\frac{2\pi}{4}\kappa m} = \sum_{m=0}^{3} x[m] \; e^{j\frac{2\pi}{4}\kappa m} = X[(-\kappa)_4]$$

Therefore $\quad DFT\{ x[(-m)_4] \} = X[(-\kappa)_4] =$

$$= \Big[X[0], \; X[3], \; X[2], \; X[1] \Big]$$

$$= \Big[1, \; -j, \; -1, \; j \Big] \qquad \sslash$$

g) $DFT\{x[(2-m)_4]\}$

Call $y[m] = x[(-m)_4]$. Then $x[(2-m)_4] = y[(m-2)_4]$
and therefore

$DFT\{x[(2-m)_4]\} = e^{-j2\frac{2\pi}{4}k} Y[k] = (-1)^k X[(-k)_4] =$

$$= [1, j, 1, j] \; /\!/$$

PROBLEM 3.5   Let $x[m]$, $m = 0, \ldots, 7$ have DFT

$$X = [1, 1-j, 1, 0, 1 \cdot 0, 1, 1+j],$$

Determine the following DFT's:

a) $y[m] = x[m] e^{j\frac{2\pi}{8}m}$, $m = 0, \ldots, 7$.

Then

$$Y[k] = X[(k-1)_8] = [1+j, 1, 1-j \; 1, 0, 1, 0, 1] \checkmark$$

b) $y[m] = x[m] \circledast \delta[m-2]$.

Since $DFT\{\delta[m-2]\} = e^{-j\frac{2\pi}{8}2k}$ $DFT\{\delta[m]\} = (-j)^k$

we have $Y[k] = [1, -1-j, -1, 0, 1, 0, -1, -1+j]$ //

Problem 3.6   Let x have DFT $X = [1, j, 1, -j]$.

Determine the following DFT's:

a) $y[m] = (-1)^m x[m] = e^{j 2 \frac{2\pi}{4} m} x[m]$

Then
$$Y[k] = DFT\{y[m]\} = X[(k-2)_4] = [1, -j, 1, j]\;\|$$

b) $y[m] = x[(m+1)_4]$

Then
$$Y[k] = e^{j \frac{2\pi}{4} k} X[k] = j^k X[k] = [1, -1, -1, -1]\;\|$$

c) $y[m] = x[m] * \delta[(m-2)_4]$

Since $DFT\{\delta[(m-2)_4]\} = e^{-j\frac{2\pi}{4}2k} = (-1)^k$, $k = 0, \ldots, 3$

Then

$$Y[k] = X[k](-1)^k = [1, -j, 1, j] \; //$$

d) $DFT\{x[(-m)_4]\} = X[(-k)_4] = [1, -j, 1, j] \; //$

Problem

Two finite sequences $x = [x[0], x[1], x[2], x[3]]$ and $h = [h[0], h[1], h[2], h[3]]$ have DFT's given by

$$X = DFT\{x\} = [1, j, -1, -j]$$
$$H = DFT\{h\} = [0, 1+j, 1, 1-j]$$

Using the properties of the DFT (do not compute x and h explicitly, compute the following:

a) DFT $\{[x[3], x[0], x[1], x[2]]\}$

b) DFT $\{[h[0], -h[1], h[2], -h[3]]\}$

c) DFT $\{h \otimes x\}$, where $\otimes$ denotes circular convolution

d) DFT $\{[x[0], h[0], x[1], h[1], x[2], h[2], x[3], h[3]]\}$

Solution

a) DFT $\{[x[3], x[0], x[1], x[2]]\} = $ DFT $\{x[(n-1)_4]\} = w_4^{-k} X[k]$ with $w_4 = e^{-j2\pi/4} = -j$ Therefore
DFT $\{x[3], x[0], x[1], x[2]\} = (-j)^k [1, j, -1, -j] = [1, 1, 1, 1]$

b) DFT $\{[h[0], -h[1], h[2], -h[3]]\} = $
$\quad$ DFT $\{(-1)^n h[n]\} = $ DFT $\{e^{-j2(2\pi/4)n} h[n]\} = H[(k-2)_4] = [1, 1-j, 0, 1+j]$

c) DFT $\{h \otimes x\} = H[k] X[k] = [0, -1+j, -1, -1-j]$

d) Let $y = [x[0], h[0], x[1], h[1], x[2], h[2], x[3], h[3]]$, with length $N = 8$. Therefore its DFT is

$$Y[k] = \sum_{n=0}^{7} y[n] w_8^{nk} = \sum_{m=0}^{3} y[2m] w_8^{2mk} + \sum_{m=0}^{3} y[2m+1] w_8^{(2m+1)k}$$

$$Y[k] = X[k] + w_8^k H[k], \quad \text{for } k = 0, \ ..., 7$$

This yields
$$Y = \left[1, \ j + (1+j) e^{-\frac{j\pi}{4}}, \ -1-j, \ -j + (1-j) e^{-\frac{3j\pi}{4}}, \ 1, \ j + (1+j) e^{\frac{3j\pi}{4}}, \ -1+j, \ -j + (1-j) e^{\frac{j\pi}{4}}\right]$$

■ **Problem 3.8**

Problem

Two finite sequences h and x have the following DFT's:

$$X = \text{DFT}\{x\} = [1, -2, 1, -2]$$
$$H = \text{DFT}\{h\} = [1, j, 1, -j]$$

Let $y = h \otimes x$ be the four point circular convolution of the two sequences. Using the properties of the DFT (do not compute $x[n]$ and $h[n]$),

a) determine DFT $\{x[(n-1)_4]\}$ and DFT $\{h[(n+2)_4]\}$;

b) determine $y[0]$ and $y[1]$.

Solution:

a) $DFT\{x[(m-1)_4]\} = e^{-j\frac{2\pi}{4}K} X[k] = -j^K X[k] =$

$$= [1, 2j, -1, -2j] /\!/$$

$DFT\{h[(m+2)_4]\} = e^{j\frac{2\pi}{4}2K} H[k] = (-1)^K H[k] =$

$$= [1, -j, 1, j] /\!/$$

b) Recall $y[m] = IDFT\{Y[u]\} = \frac{1}{N}\sum_{k=0}^{N-1} Y[k] e^{jk\frac{2\pi}{N}m}$

Then

$$y[0] = \frac{1}{4}\left(Y[0] + Y[1] + Y[2] + Y[3]\right) =$$

$$= \frac{1}{4}\left(H[0]X[0] + H[1]X[1] + H[2]X[2] + H[3]X[3]\right)$$

$$= \frac{1}{4}\left(1 - 2j + 1 + 2j\right) = \frac{1}{2}$$

$$y[1] = \frac{1}{4}\left(1 + (-2j)j + j^2 + (2j)j^3\right) = \frac{1}{4}\left(1 + 2 - 1 + 2\right) = 1 \quad //$$

■ Problem 3.9

Problem

Let x be a finite sequence with DFT

$$X = \text{DFT}\{x\} = [0, \, 1 + j, \, 1, \, 1 - j]$$

Using the properties of the DFT determine the DFT's of the following:

a) $y[n] = e^{j(\pi/2)n} x[n]$

b) $y[n] = \cos(\frac{\pi}{2} n) x[n]$

c) $y[n] = x[(n - 1)_4]$

d) $y[n] = [0, 0, 1, 0] \oplus x[n]$ with $\oplus$ denoting circular convolution

Solution

a) Since $e^{j(\pi/2)n} x[n] = e^{j(2\pi/4)n} x[n]$ then DFT$\{e^{j(\pi/2)n} x[n]\} = X[(k - 1)_4] = [1 - j, 0, 1 + j, 1]$

b) In this case $y[n] = \frac{1}{2} e^{j(2\pi/4)n} x[n] + \frac{1}{2} e^{-j(2\pi/4)n} x[n]$ and therefore its DFT is
$\frac{1}{2} X[(k - 1)_4] + \frac{1}{2} X[(k + 1)_4] = \frac{1}{2}[1 - j, 0, 1 + j, 1] + \frac{1}{2}[1 + j, 1, 1 - j, 0]$. Putting things together we obtain

$$Y = [1, \, \tfrac{1}{2}, \, 1, \, \tfrac{1}{2}]$$

c) DFT$\{x[(n - 1)_4] = e^{-j(2\pi/4)k} X[k] = (-j)^k X[k]$ and therefore

$$Y = [0, \, 1 - j, \, -1, \, 1 + j]$$

d) DFT$\{x[(n - 2)_4]\} = (-j)^{2k} X[k] = (-1)^k X[k] = [0, \, -1, \, j, \, 1, \, -1 + j]$

Problems on DFT: Manipulation of Properties and Derivation of Other Properties

■ **Problem 3.11**

Problem

You know that DFT $\{[1, 2, 3, 4]\} = [10, \ -2 + 2j, \ -2, \ -2 - 2j]$. Use the minimum number of operations to compute the following transforms:

Solution

a) Let

$$x = [1, 2, 3, 4]$$
$$s = [1, 2, 3, 4, 1, 2, 3, 4]$$
$$y = [1, 2, 3, 4, 0, 0, 0, 0]$$

Then we can write $y[n] = s[n]\, w[n]$, $n = 0, \ ..., 7$ where

$$w = [1, 1, 1, 1, 0, 0, 0, 0]$$

and then, by a property of the DFT we can determine

$$Y[k] = \text{DFT}\{y[n]\} = \text{DFT}\{s[n]\, w[n]\} = \tfrac{1}{N}\, W[k] \otimes S[k]$$

Let's relate the respective DFT's:

$$S[k] = \text{DFT}\{s\} = \sum_{n=0}^{7} s[n]\, w_8^{nk}$$

$$= \sum_{n=0}^{3} s[n]\, w_8^{nk} + \sum_{n=0}^{3} s[n+4]\, w_8^{(n+4)k}$$

$$= \sum_{n=0}^{3} s[n]\, w_8^{nk} + (-1)^k \sum_{n=0}^{3} s[n]\, w_8^{nk}$$

$$= (1 + (-1)^k) \sum_{n=0}^{3} s[n]\, w_8^{nk}$$

This implies that

$$S[2\,k] = 2\,X[k]$$
$$S[2\,k + 1] = 0$$

and therefore

$$S = \text{DFT}\,\{[1, 2, 3, 4, 1, 2, 3, 4]\} = 2\,[X[0], 0, X[1], 0, X[2], 0, X[3], 0]$$

Furthermore:

$$W = \text{DFT}\,\{w\} = \sum_{n=0}^{3} w_8^{n\,k} = \begin{cases} \frac{1-(-1)^k}{1-w_8^k} & \text{if } k = 1, \,...,\, 7 \\ 4 & \text{if } k = 0 \end{cases}$$

and therefore

$$W = [4,\ 1 - j2.4142,\ 0,\ 1 - j\,0.4142,\ 0,\ 1 + j0.4142,\ 0,\ 1 + j\,2.4142]$$

Finally the result:

$$Y[k] = \frac{1}{8} \sum_{m=0}^{7} S[m]\,W[(k - m)_8]$$

$$= \frac{1}{8} \sum_{p=0}^{3} 2\,X[p]\,W[(k - 2\,p)_8]$$

$$= \frac{1}{4} \sum_{p=0}^{3} X[p]\,W[(k - 2\,p)_8]$$

where we used the fact that $S[m] \neq 0$ only for m even, and $S[2\,p] = 2\,X[p]$.

The final result is simpler than it seems. In fact notice that $W[2] = W[4] = W[6] = 0$. This implies that

$$Y[0] = \frac{1}{4}\,W[0]\,X[0] = \frac{1}{2}\,X[0] = 10$$
$$Y[2] = \frac{1}{4}\,W[0]\,X[1] = \frac{1}{2}\,X[1] = -2 + 2\,j$$
$$Y[4] = \frac{1}{4}\,W[0]\,X[2] = \frac{1}{2}\,X[2] = -2$$

and we have only to compute

$$Y[1] = \frac{1}{4}\,(W[1]\,X[0] + W[7]\,X[1] + W[5]\,X[2] + W[3]\,X[3]) = -0.4142 - j\,2.4142$$
$$Y[3] = \frac{1}{4}\,(W[3]\,X[0] + W[1]\,X[1] + W[7]\,X[2] + W[5]\,X[3]) = 2.4142 - j\,1.2426$$

All other terms are computed by symmetry as $Y[8 - k] = Y^*[k]$, for $k = 1, 2, 3$.

b) Let $y = [1, 0, 2, 0, 3, 0, 4, 0]$. Then we can write

$$Y[k] = \sum_{n=0}^{7} y[n]\,w_8^{n\,k} = \sum_{m=0}^{3} x[m]\,w_8^{2\,m\,k}$$

since $y[2\,m] = x[m]$ and $y[2\,m + 1] = 0$. From the fact that

$$w_8^2 = e^{-j\frac{2\pi}{8}\,2} = w_4$$

we can write

$$Y[k] = \sum_{m=0}^{3} x[m]\, w_4^{m\,k} = X[k],\ k = 0,\ 1,\ ...,\ 7$$

and therefore

$$Y = [10,\ -2 + 2\,j,\ -2,\ -2 - 2\,j,\ 10,\ -2 + 2\,j,\ -2,\ -2 - 2\,j]$$

■ Problem 3.12

Problem

You know that the DFT of the sequence `x = [x[0], ..., x[N - 1]]` is
`X = [X[0], ..., X[N - 1]]`. Now consider two sequences:

`s = [x[0], ..., x[N - 1], x[0], ..., x[N - 1]]`

$y = \left[x[0],\ ...,\ x[N-1],\ \underbrace{0,\ ...,\ 0}_{N\,times}\right]$

both of length `2 N`. Let `S = DFT {s}` and `Y = DFT[y]`.

a) Show that `S[2 m] = 2 X[m]`, and `S[2 m + 1] = 0`, for `m = 0, ..., N - 1`;

b) Show that `Y[2 m] = X[m]` for `m = 0, ..., N - 1`;

c) Determine an expression for `Y[2 m + 1], k = 0, ..., N - 1`. Use the fact that
`y[n] = s[n] w[n]` where $w = \left[\underbrace{1,\ ...,\ 1}_{N\,times},\ \underbrace{0,\ ...,\ 0}_{N\,times}\right]$ is the rectangular sequence.

Solution

a) $S[k] = \sum_{n=0}^{N-1} x[n]\, (w_{2N})^{nk} + \sum_{n=0}^{N-1} x[n]\, (w_{2N})^{(n+N)\,k}$ from the definition of `s[n]`. Now
consider that $w_{2N}^{Nk} = e^{-j\,(2\pi/2N)\,2N} = -1$. Also consider that $w_{2N} = e^{-(j2\pi/2N)} = (w_N)^{\frac{1}{2}}$. Therefore
we obtain

$$S[k] = (1 + (-1)^k) \sum_{n=0}^{N-1} x[n]\, (w_{2N})^{nk} = \begin{cases} 0 & \text{if k is odd} \\ 2\,X[\frac{k}{2}] & \text{if k is even} \end{cases}$$

b) $Y[2 m] = \sum_{n=0}^{N-1} x[n]\, (w_{2N})^{2\,m\,n}$. But again $(w_{2N})^2 = w_N$ and therefore `Y[2 m] = X[m]`, for
`m = 0, ..., N - 1`

c) Since clearly `y[n] = s[n] w[n]` for all `n = 0, ..., 2 N - 1`, we can use the property of the
DFT which states

$$DFT \{s[n]\, w[n]\} = \frac{1}{2N}\, S[k] \otimes W[k]$$

where $W[k] = DFT\{w[n]\} = \sum_{n=0}^{N-1} (w_{2N})^{nk} = \frac{1-(w_{2N})^{Nk}}{1-(w_{2N})^{k}}$. Therefore

$$Y[k] = \frac{1}{2N} \sum_{m=0}^{N-1} X[m] H[(k - 2m)_{2N}]$$

where we take into account the result in part a). ie $S[2m] = X[m]$ and $S[2m+1] = 0$.

■ Problem 3.13

Problem

A problem in many software packages is that you have access only to positive indexes. Suppose you want to determine the DFT of the sequence $x = \left[0, 1, 2, 3, \underset{\underset{n=0}{\uparrow}}{4}, 5, 6, 7\right]$ with the zero index as shown, and the DFT algorithm let you compute the DFT of a sequence like $x[0], \dots, x[N-1]$, what do you do?

Solution

Since x is effectively a period of a periodic sequence, we determine the DFT of one period starting at $n = 0$, which yields

$$X = DFT\{[4, 5, 6, 7, 0, 1, 2, 3]\}$$

■ Problem 3.14

Problem

Let $X[k] = DFT\{x[n]\}$ with $n, k = 0, \dots, N-1$. Determine the relationships between $X[k]$ and the following DFT's:

a) $DFT\{x^*[n]\}$

b) $DFT\{x[(-n)_N]\}$

c) $DFT\{Re\{x[n]\}\}$

d) $DFT\{Im\{x[n]\}\}$

e) apply all the above properties to the sequence $x = IDFT\{[1, -j, 2, 3j]\}$

Solution

a) $DFT\{x^*[n]\} = \sum_{n=0}^{N-1} x^*[n] w_N^{nk} = \left(\sum_{n=0}^{N-1} x[n] w_N^{-nk}\right)^* = X^*[(-k)_N]$

b) $DFT\{x[(-n)_N]\} = \sum_{n=0}^{N-1} x[(-n)_N] w_N^{nk} = \sum_{n=0}^{N-1} x[n] w_N^{-nk} = X[(-k)_N]$

c) $DFT\{Re\{x[n]\}\} = \frac{1}{2} DFT\{x[n]\} + \frac{1}{2} DFT\{x^*[n]\} = \frac{1}{2} X[k] + \frac{1}{2} X^*[(-k)_N]$

d) $\text{DFT}\{\text{Im}\{x[n]\}\} = \frac{1}{2j} \text{DFT}\{x[n]\} - \frac{1}{2j} \text{DFT}\{x^*[n]\} = \frac{1}{2j} X[k] - \frac{1}{2j} X^*[(-k)_N]$

e) $\text{DFT}\{x^*[n]\} = [1, -3j, 2, j]$

$\quad \text{DFT}\{x[(-n)_N]\} = [1, 3j, 2, -j]$

$\quad \text{DFT}\{\text{Re}\{x[n]\}\} = [1, -2j, 2, 2j]$

$\quad \text{DFT}\{\text{Im}\{x[n]\}\} = [0, 1, 0, 1]$

■ Problem 3.15

Problem

Let $x[n] = \text{IDFT}\{X[k]\}$ for n, $k = 0$, ..., $N-1$. Determine the relationship between $x[n]$ and the following IDFT's:

a) $\text{IDFT}\{X^*[k]\}$

b) $\text{IDFT}\{X[(-k)_N]\}$

c) $\text{IDFT}\{\text{Re}\{X[k]\}\}$

d) $\text{IDFT}\{\text{Im}\{X[k]\}\}$

e) apply all the above properties to the sequence $X[k] = \text{DFT}\{[1, -2j, +j, -4j]\}$

Solution

a) $\text{IDFT}\{X^*[k]\} = \frac{1}{N} \sum_{k=0}^{N-1} X^*[k]\, w_N^{-nk} = \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k]\, w_N^{nk} \right)^* = x^*[(-n)_N]$

b) $\text{IDFT}\{X[(-k)_N]\} = \frac{1}{N} \sum_{k=0}^{N-1} X[(-k)_N]\, w_N^{-nk} = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, w_N^{nk} = x[(-n)_N]$

c) $\text{IDFT}\{\text{Re}\{X[k]\}\} = \frac{1}{2} \text{IDFT}\{X[k]\} + \frac{1}{2} \text{IDFT}\{X^*[k]\} = \frac{1}{2} x[n] + \frac{1}{2} x^*[(-n)_N]$

d) $\text{IDFT}\{\text{Im}\{X[k]\}\} =$
$\quad \frac{1}{2j} \text{IDFT}\{X[k]\} - \frac{1}{2j} \text{IDFT}\{X^*[k]\} = \frac{1}{2j} x[n] - \frac{1}{2j} x^*[(-n)_N]$

e) $\text{IDFT}\{X^*[k]\} = [1, 4j, -j, 2j]$

$\quad \text{IDFT}\{X[(-k)_N]\} = [1, -4j, j, -2j]$

$\quad \text{IDFT}\{\text{Re}\{X[k]\}\} = [1, j, 0, -j]$

$\quad \text{IDFT}\{\text{Im}\{X[k]\}\} = [0, -3, 1, -3]$

■ Problem 3.16

Problem

Let $x[n]$ be an infinite sequence, periodic with period N. This sequence is the input to a BIBO stable LTI system with impulse response $h[n]$, $-\infty < n < +\infty$. Say how you can use the DFT to determine the output of the system.

Solution

Call $x = [x[0], \ldots, x[N-1]]$ one period of the signal, and let $X = \text{DFT}\{x\}$. Then the whole infinite sequence $x[n]$ can be expressed as $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{k2\pi}{N}n}$. Therefore the output becomes

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, H\left(\frac{k2\pi}{N}\right) e^{j\frac{k2\pi}{N}n}$$

We can see that the output sequence $y[n]$ is also periodic, with the same period N given by

$$y[n] = \text{IDFT}\left\{H\left(\frac{k2\pi}{N}\right) X[k]\right\}, \text{for } n, k = 0, \ldots, N-1$$

■ Problem 3.17

Problem

Let $x[n]$ be a periodic signal with one period given by $\left[1, -2, \underset{\underset{n=0}{\uparrow}}{3}, -4, 5, -6\right]$ with the zero index as shown.

It is the input to a LTI system with impulse response $h[n] = 0.8^{|n|}$. Determine one period of the output sequence $y[n]$.

Solution

The input signal is periodic with period $N = 6$ and it can be written as

$$x[n] = \frac{1}{6} \sum_{k=0}^{5} X[k]\, e^{jk\frac{\pi}{3}n}, \text{ for all } n,$$

where

$$X[k] = \text{DFT}\{[3, -4, 5, -6, 1, -2]\} =$$
$$[-3.0, \; 3.0 + j\,1.7321, \; -3.0 - j\,5.1962, \; 21.0, \; -3.0 + j\,5.1962, \; 3.0 - j\,1.7321]$$

The frequency response of the system is given by

$$H(\omega) = \text{DTFT}\{0.8^n u[n] + 1.25^n u[-n-1]\} = \frac{e^{j\omega}}{e^{j\omega} - 0.8} - \frac{e^{j\omega}}{e^{j\omega} - 1.25}$$

Therefore the response to the periodic signal $x[n]$ becomes

$$y[n] = \frac{1}{6} \sum_{k=0}^{5} H[k] X[k] e^{jk\frac{\pi}{3}n}, \text{ for all } n,$$

with

$$H[k] = H(\omega)\big|_{\omega=k\pi/3} = \frac{e^{jk\frac{\pi}{3}}}{e^{jk\frac{\pi}{3}}-0.8} - \frac{e^{jk\frac{\pi}{3}}}{e^{jk\frac{\pi}{3}}-1.25}, k = 0, \ldots, 5$$

Therefore one period of the output signal is determined as

$$y[n] = \text{IDFT}\{H[k] X[k]\}, n = 0, \ldots, 5,$$

where

$$H[k] = [9.0, \ 0.4286, \ 0.1475, \ 0.1111, \ 0.1475, \ 0.4286]$$

Computing the IDFT we obtain

$$y[n] = [-3.8301, \ -4.6079, \ -3.8160, \ -5.4650, \ -4.6872, \ -4.5938], n = 0, \ldots, 5$$

The DFT: Data Analysis

■ Problem 3.18

Problem

A narrowband signal is sampled at 8 kHz and we take the DFT of 16 points as follows. Determine the best estimate of the frequency of the sinusoid, its possible range of values and an estimate of the amplitude:

```
X = [0.4889, 4.0267 - j24.6698, 2.0054 - j5.0782,
   1.8607 - j2.8478, 1.8170 - j1.8421, 1.7983 - j1.2136,
 1.7893 - j0.7471, 1.7849 - j0.3576, 1.7837, 1.7849 + j0.3576,
   1.7893 + j0.7471, 1.7983 + j1.2136, 1.8170 + j1.842,
   1.8607 + j2.8478, 2.0054 + j5.0782, 4.0267 + j24.6698]
```

Solution

Looking at the DFT we see that there is a peak for $k = 1$ and $k = 15$, withe the respective values being $X[1] = 4.0267 - j24.6698$, and $X[15] = 4.0267 + j24.6698$. The magnitude is $|X[1]| = |X[15]| = 25.0259$. Therefore the frequency is within the interval

$$(k_0 - 1) \frac{F_s}{N} < F < (k_0 + 1) \frac{F_s}{N}$$

with $k_0 = 1$. $F_s = 8$ kHz and $N = 16$. This yield an estimated frequency $0 < F < 2 \times \frac{8}{16}$ kHz $= 1$ kHz.

■ Problem 3.19

Problem

A real signal $x(t)$ is sampled at 8 kHz and we store 256 samples $x[0], \ldots, x[255]$. The magnitude of the DFT $X[k]$ has two sharp peaks at $k = 15$ and $k = 241$. What can you say about the signal?

Solution

The signal has a dominant frequency component in the range $14 \times \frac{8}{256} < F < 16 \times \frac{8}{256}$ kHz, that is to say 0.4375 kHz $< F < 0.500$ kHz.

■ Problem 3.21

Problem

We have seen in the theory that the frequency resolution with the DFT is $\triangle F = 2 F_s / N$, with F_s the sampling frequency and N the data length. Since the lower $\triangle F$ the better, by lowering the sampling frequency F_s, while keeping the data length N constant, we get arbitrary good resolution! Do you buy that? Why? or Why not?

Solution

It is true that by lowering the sampling frequency F_s and keeping the number of points N constant we improve the resolution. In fact by so doing the sampling interval T_s gets longer and we get a longer interval of data. However we cannot decrease the sampling frequency F_s indefinitely, since we have to keep it above twice the bandwidth of the signal to prevent aliasing.

■ **Problem 3.23**

We want to plot the DTFT of a sequence which is not in the tables. Using the DFT and an increasing number of points sketch a plot of the DTFT (magnitude and phase) of each of the following sequences:

Solution

For every signal in this problem we need to take the DFT of the following sequence

$$x_N = [x[0], \ ..., \ x[N-1], \ x[-N], \ ..., \ x[-1]]$$

of length $2\,N$. In other words the first N points correspond to positive indexes, while the last N points correspond to negative indexes. By increasing the value of the parameter N we observe the convergence of the DFT.

a) $x[n] \ = \ \frac{1}{n} \ u[n-1]$

The following figures show the DFT for different lengths $N = 2^7, \ 2^9, \ 2^{11}$ (magnitude and phase):

$$N = 2^7$$

$$N = 2^9$$
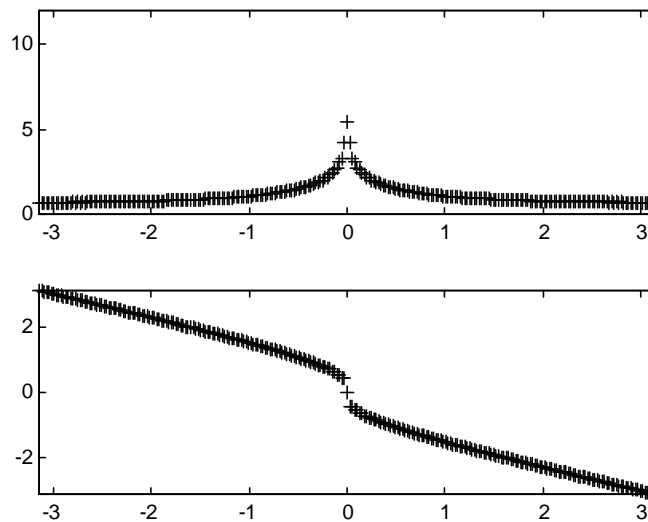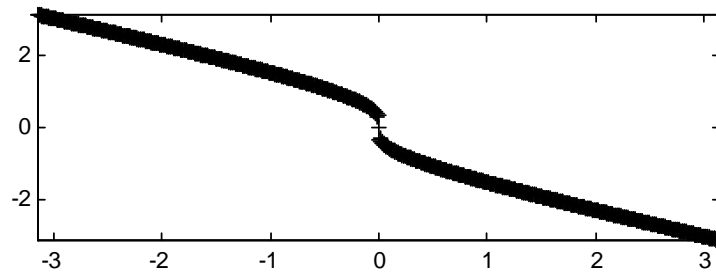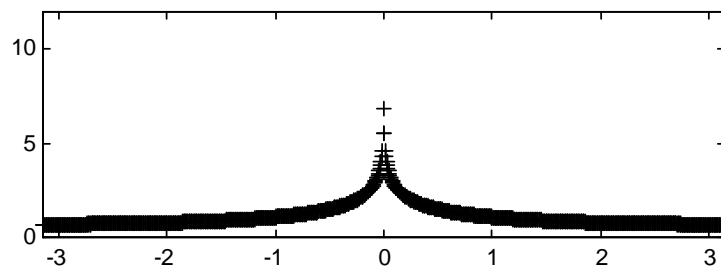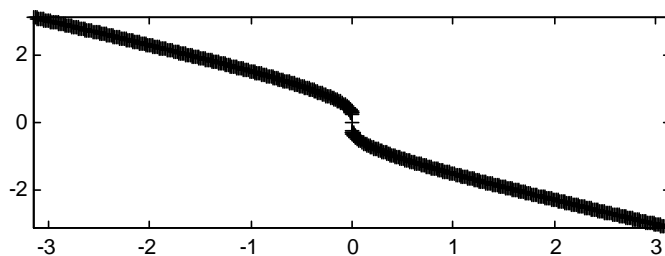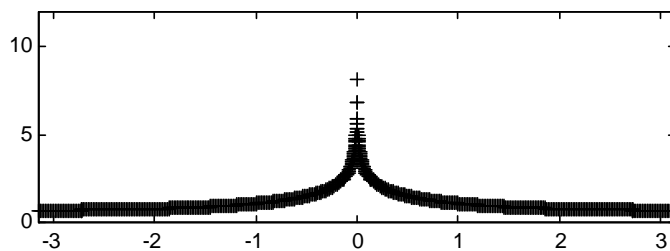
$$N = 2^{11}$$

Notice that at $\omega = 0$ the DFT does not converge (the magnitude increases with N) while it converges at all other frequencies. As a consequence, we can say that the DFT converges to the DTFT for all $\omega \neq 0$. This is expected, since the signal $x[n]$ is NOT absolutely summable.
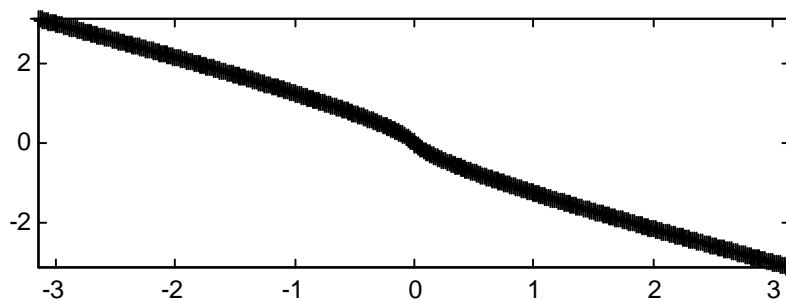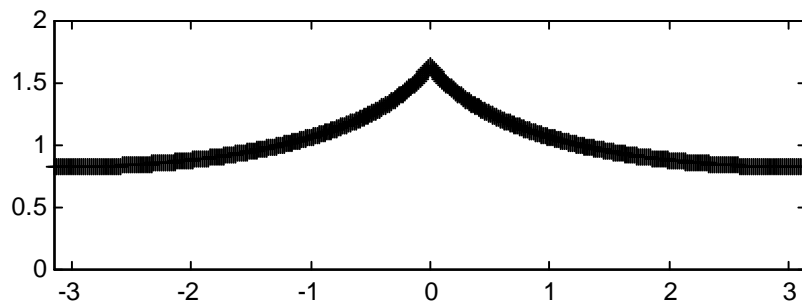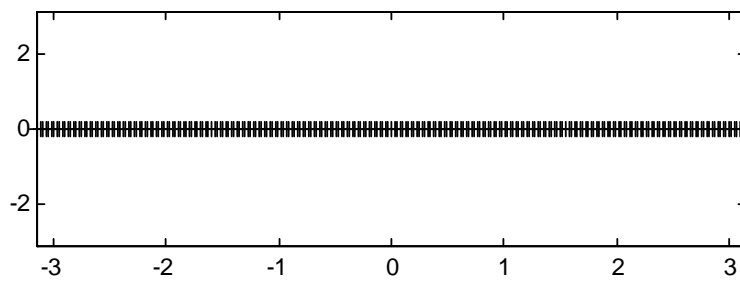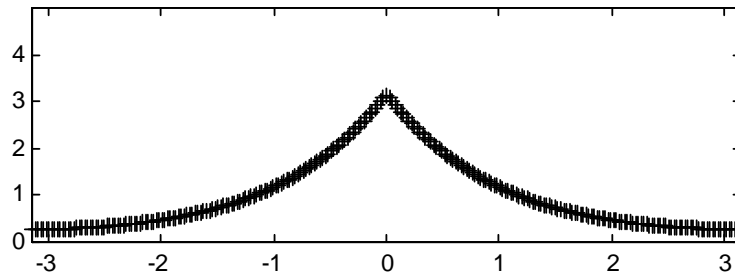
Also notice the discontinuity in the phase at $\omega = 0$.

b) $x[n] = \frac{1}{n^2} u[n-1]$

The following figures show the DFT for different lengths $N = 2^7,\ 2^{11}$ (magnitude and phase):
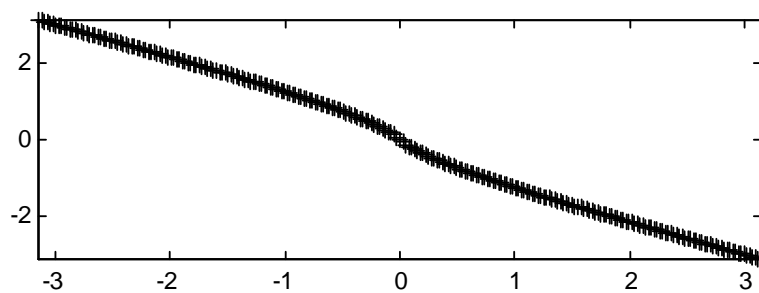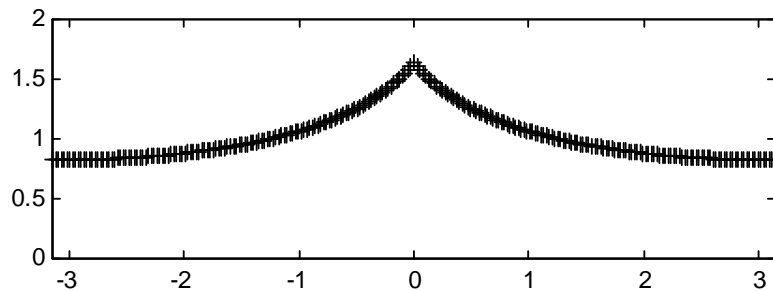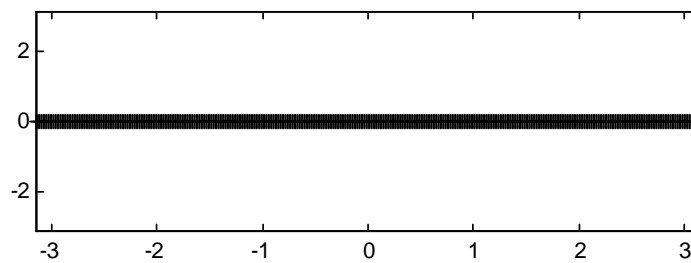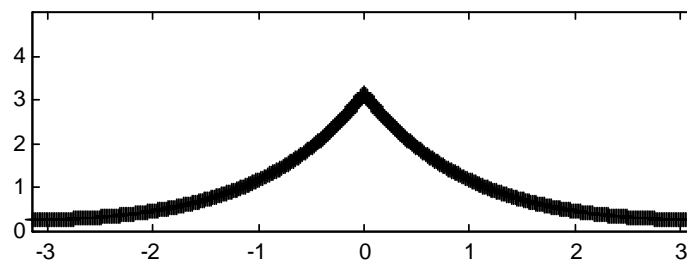

$$N = 2^7$$

$$N = 2^{11}$$

Notice that the DFT converges everywhere, as expected since the signal is absolutely summable.

c) $x[n] = \frac{1}{n^2+1}$

The following figures show the DFT for different lengths $N = 2^7, \ 2^{11}$ (magnitude and phase):
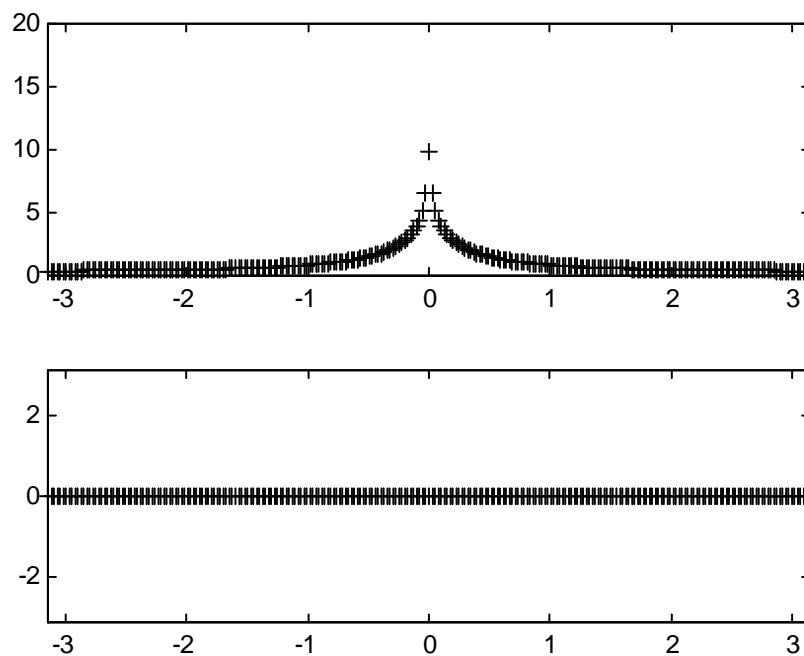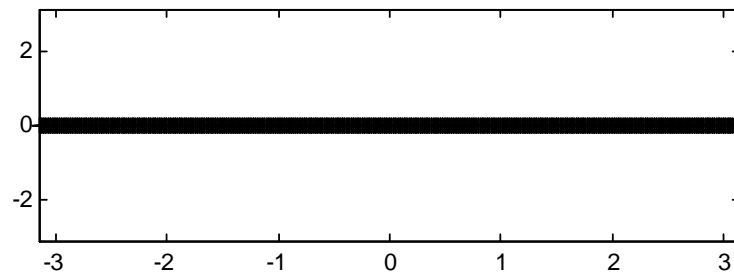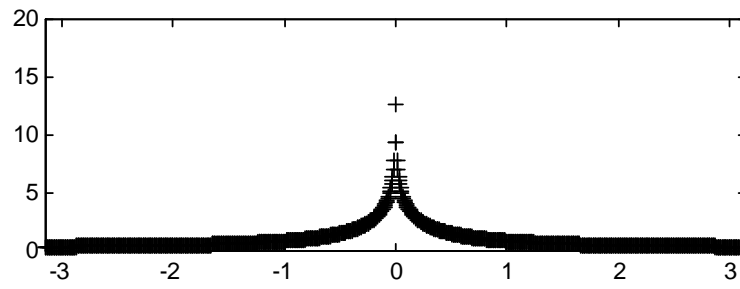

$$N = 2^7$$

$$N = 2^{11}$$

Notice that the DFT converges everywhere, as expected since the signal is absolutely summable.
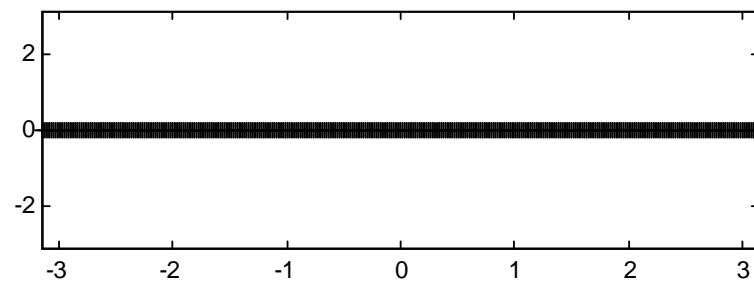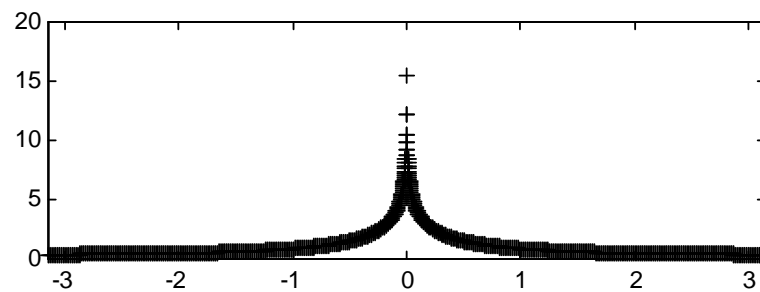
d) $x[n] = \frac{1}{|n|+1}$

The following figures show the DFT for different lengths $N = 2^7,\ 2^9,\ 2^{11}$ (magnitude and phase):

$$N = 2^7$$

$$N = 2^9$$

$$N = 2^{11}$$

Notice that at $\omega = 0$ the DFT does not converge (the magnitude increases with N) while it converges at all other frequencies. As a consequence, we can say that the DFT converges to the DTFT for all $\omega \neq 0$. This is expected, since the signal $x[n]$ is NOT absolutely summable.

Review Problems

■ Problem 3.26

Problem.

In the system shown, let the continuous time signal x (t) have Fourier Transform as shown. Sketch X (ω) = DTFT {x[n]} for the given sampling frequency.
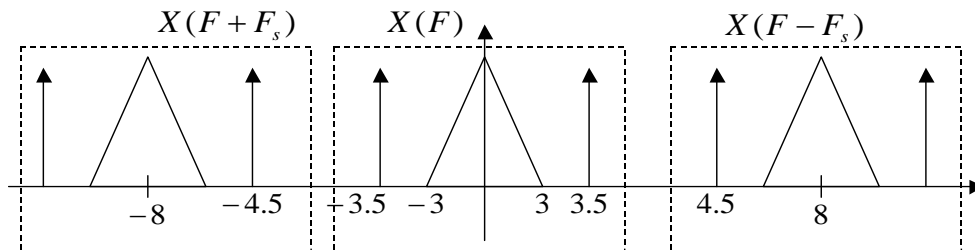
Solution.

Recall that, for a sampled signal,

$$\text{DTFT } \{\text{x[n]}\} = \text{F}_\text{s} \sum_{\text{k}=-\infty}^{+\infty} \text{X (F} - \text{kF}_\text{s}) \Big|_{\text{F}=\omega\text{F}_\text{s}/2\,\pi}$$

with x[n] = x (nT$_\text{s}$), and T$_\text{s}$ = 1 / F$_\text{s}$.

a) F$_\text{s}$ = 8 kHz. Then clearly there is no aliasing, and therefore

we can write

$$\text{X } (\omega) = \text{F}_\text{s} \text{ X (F) } |_{\text{F}=\omega\text{F}_\text{s}/2\,\pi}, \text{ for } -\pi < \omega \leq \pi$$
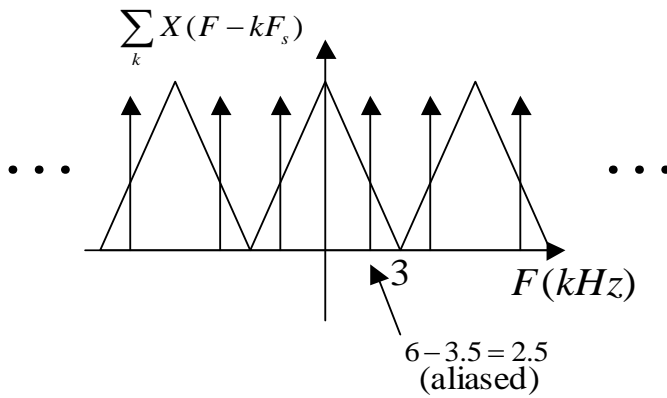
In other words we just need to rescale the frequency axis (F $\rightarrow$ ω = 2 πF / F$_\text{s}$) and the vertical axis (multiply by F$_\text{s}$), as shown below. For the "delta" function recall that the value associated is not the amplitude but it is the area. This implies

$$\delta \text{ (F } - \text{F}_0) \ |_{\text{F}=\omega\text{F}_\text{s}/2\,\pi} = \delta \ (\tfrac{\text{F}_\text{s}}{2\,\pi} \ (\omega - \omega_0)) = \tfrac{2\,\pi}{\text{F}_\text{s}} \ \delta \ (\omega - \omega_0)$$
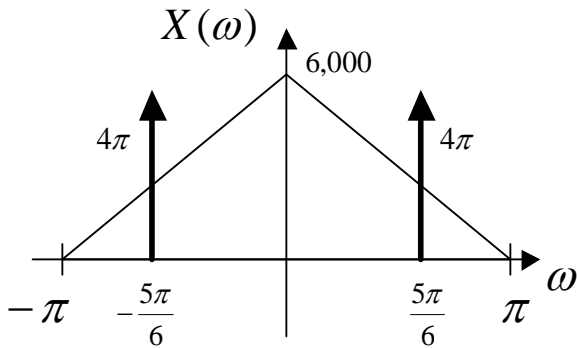
where ω_0 = 2 πF$_0$ / F$_\text{s}$.
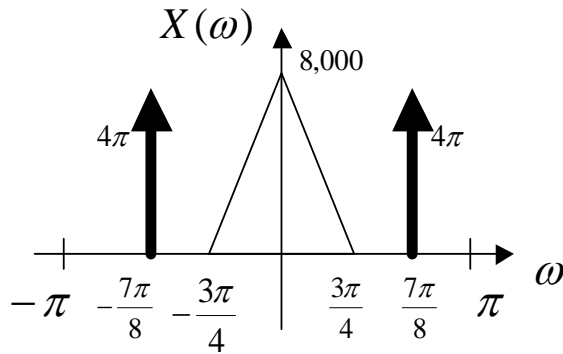
b) $F_s = 6 \text{ kHz}$. In this case the two "delta" functions are aliased, as shown below

The aliased frequency is at 6-3.5=2.5kHz. The DTFT of the sampled signal is then given in the figure below.

c) $F_s = 2 \text{ kHz}$. In this case the whole signal is aliased, not only the narrowband component as in the previous case. In order to make the figure not too confusing, consider the parts of the signal separately.

For the "broadband" component of the signal, the repetition in frequency yields the following:
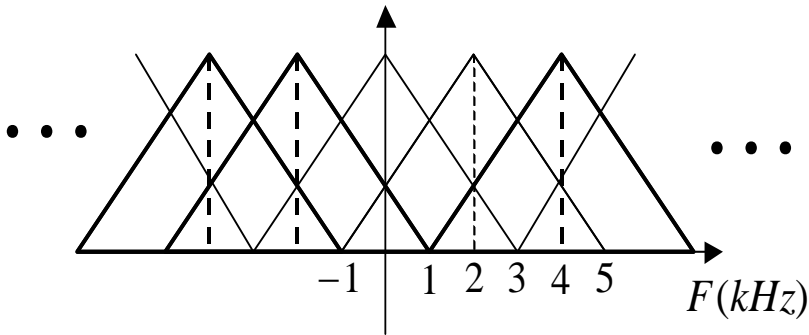
Then summing all components we obtain the result shown. Just sum within one period (say $-1 \text{ kHz} < F \leq 1 \text{ kHz}$) and repeat periodically.
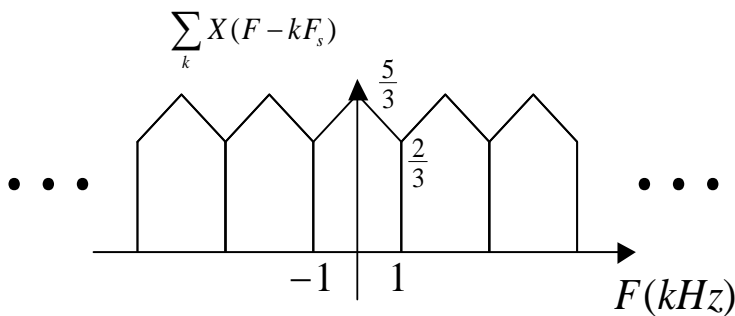
For the "narrowband" component (the two "delta" functions) we obtain the following:
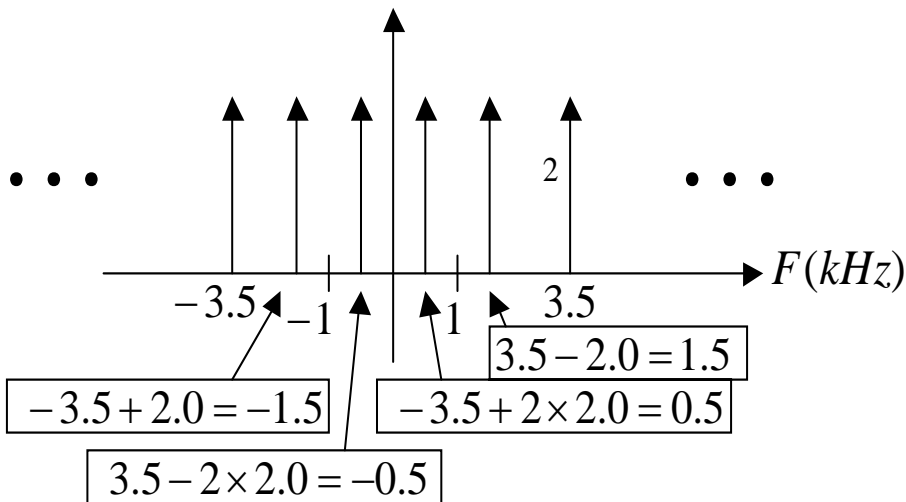
Finally consider only the interval $-1 \text{ kHz} \leq F < 1 \text{ kHz}$, put the two together and properly rescale the axis ($\times F_s$ the vertical axis, and $\times 2\pi / F_s$ the horizontal axis) to obtain $X(\omega)$:

■ Problem 3.27

Problem.

The signal $x(t) = 3\cos(2\pi F_1 t + 0.25\pi) - 2\sin(2\pi F_2 t - 0.3\pi)$ has frequencies $F_1 = 3\text{ kHz}$ and $F_2 = 4\text{ kHz}$. Determine the DTFT of the sampled sequence for the given sampling frequencies:

Solution.

The continuous time signal can be written in terms of complex exponentials as

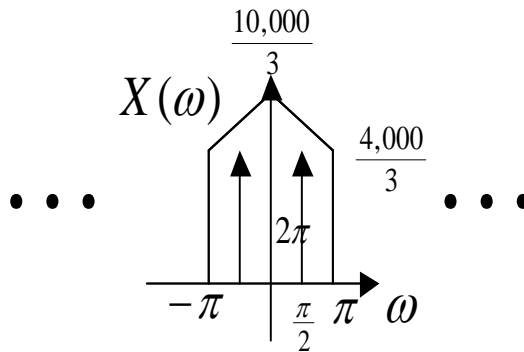$$x(t) = \frac{3}{2}e^{j0.25\pi}e^{j2\pi F_1 t} + \frac{3}{2}e^{-j0.25\pi}e^{-j2\pi F_1 t} + $$
$$+ je^{-j0.3\pi}e^{j2\pi F_2 t} - je^{j0.3\pi}e^{-j2\pi F_2 t}$$

Therefore its Fourier Transform becomes

$$X(F) = \frac{3}{2}e^{j0.25\pi}\delta(F - F_1) + \frac{3}{2}e^{-j0.25\pi}\delta(F + F_1) + $$
$$+ e^{j0.2\pi}\delta(F - F_2) + e^{-j0.2\pi}\delta(F + F_2)$$

which is shown below.
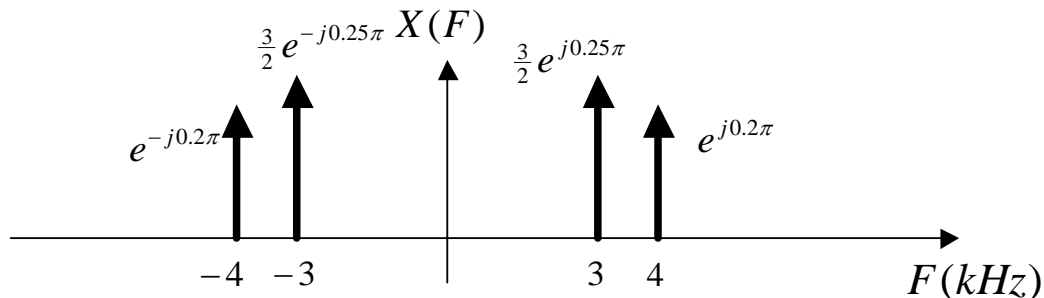
a) $F_s = 9\text{ kHz}$. In this case there is no aliasing, and it is just a matter of computing $X(\omega) = F_s X(F)\ |_{F = \omega F_s / 2\pi}$ shown below.

$$X(\omega)$$

$$3\pi e^{-j0.25\pi} \qquad 3\pi e^{j0.25\pi}$$

$$2\pi e^{-j0.2\pi} \qquad\qquad\qquad 2\pi e^{j0.2\pi}$$

$$-\pi \quad -\frac{8\pi}{9} \quad -\frac{2\pi}{3} \qquad\qquad \frac{2\pi}{3} \quad \frac{8\pi}{9} \quad \pi \qquad \omega$$

b) $F_s = 7$ kHz. In this case there is aliasing as shown:

$$X(F)$$

$$\frac{3}{2} e^{-j0.25\pi} \qquad \frac{3}{2} e^{j0.25\pi}$$

$$e^{-j0.2\pi} \qquad\qquad\qquad e^{j0.2\pi}$$

$$-4 \quad -3 \qquad\qquad 3 \quad 4 \qquad F(kHz)$$
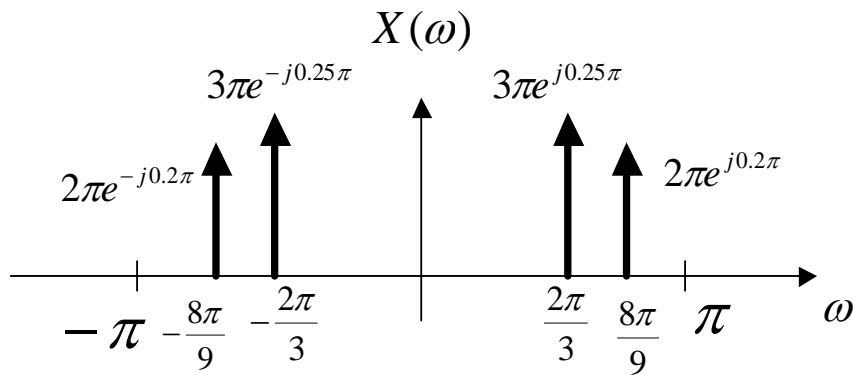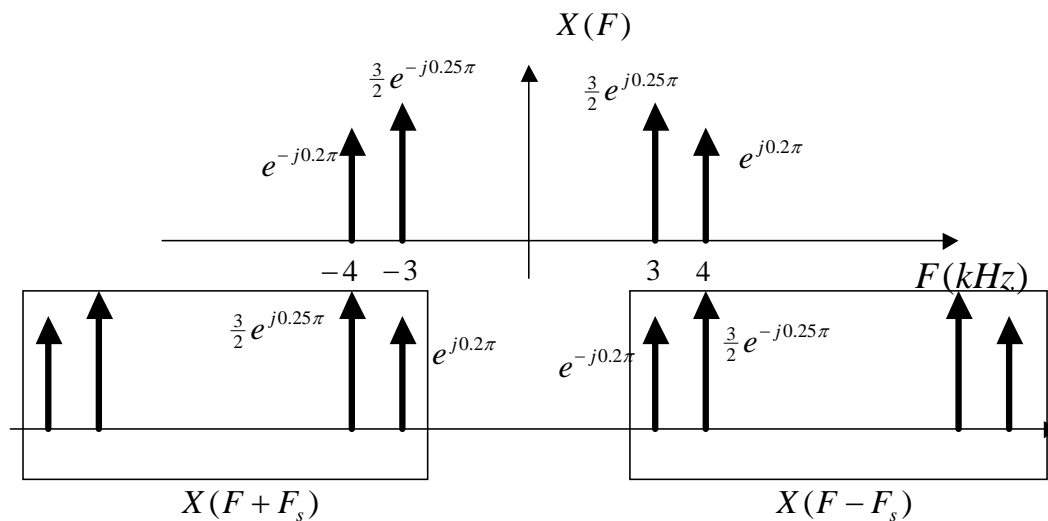
$$\frac{3}{2} e^{j0.25\pi} \qquad e^{j0.2\pi} \qquad e^{-j0.2\pi} \qquad \frac{3}{2} e^{-j0.25\pi}$$

$$X(F+F_s) \qquad\qquad\qquad X(F-F_s)$$

Summing the components within the interval -3.5 kHz $\leq F < 3.5$ kHz we obtain only two "delta" functions as

$$\sum_k X(F - kF_s) = \left(e^{-j0.2\pi} + \frac{3}{2} e^{j0.25\pi}\right) \delta(F - 3,000) +$$
$$+ \left(e^{j0.2\pi} + \frac{3}{2} e^{-j0.25\pi}\right) \delta(F + 3,000)$$

Therefore

$$X(\omega) = 2\pi \times 1.9285 \, e^{j0.2477} \, \delta\left(\omega - \frac{6\pi}{7}\right) + 2\pi \times 1.9285 \, e^{-j0.2477} \, \delta\left(\omega + \frac{6\pi}{7}\right)$$

c) $F_s = 5$ kHz. The aliased components are computed from the figure below.

$$X(F)$$
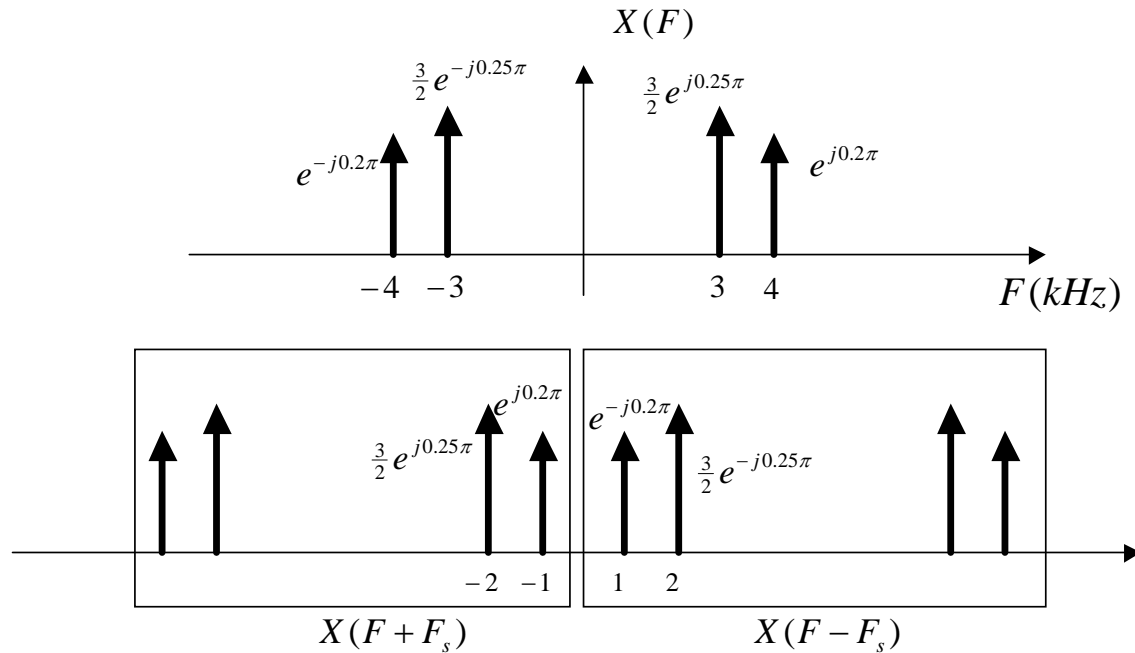
Therefore within the interval -2.5 kHz $\leq F < 2.5$ kHz we can write

$$\sum_k X\ (F - kF_s) = \frac{3}{2}\ e^{j0.25\pi}\ \delta\ (F + 2000)\ +\ e^{j0.2\pi}\ \delta\ (F + 1000)\ +$$
$$+\ \frac{3}{2}\ e^{-j0.25\pi}\ \delta\ (F - 2000)\ +\ e^{-j0.2\pi}\ \delta\ (F - 1000)$$

Therefore

$$X\ (\omega)\ =\ 3\ \pi e^{j0.25\pi}\ \delta\ (\omega + \frac{4\pi}{5})\ +\ 2\ \pi e^{j0.2\pi}\ \delta\ (\omega + \frac{2\pi}{5})\ +$$
$$+\ 3\ \pi e^{-j0.25\pi}\ \delta\ (\omega - \frac{4\pi}{5})\ +\ 2\ \pi e^{-j0.2\pi}\ \delta\ (\omega - \frac{2\pi}{5})$$

for $-\pi \leq \omega < \pi$.

■ Problem 3.28

Problem.

You want to compute the DCT of a finite set of data., but you have only the DFT. Can you still compute it?

Solution.

From the definition of DCT-II,

$$X^{II}[k]\ =\ \sqrt{\frac{2}{N}}\ C[k]\ \sum_{k=0}^{N-1}\ x[n]\ \cos\ (\ \frac{\pi k\ (2\ n+1)}{2\ N}\ )$$

Expanding the cosine term we obtain

$$\sum_{k=0}^{N-1} \text{x[n] cos} \left(\frac{\pi k \, (2 \, n+1)}{2 \, N} \right) = \frac{1}{2} \, e^{j \, \frac{k\pi}{2N}} \sum_{k=0}^{N-1} \text{x[n]} \, e^{jk \, \frac{2\pi}{2N} \, n} + \frac{1}{2} \, e^{-j \, \frac{k\pi}{2N}} \sum_{k=0}^{N-1} \text{x[n]} \, e^{-jk \, \frac{2\pi}{2N} \, n}$$

Now form the sequence

$$\text{x}_0 = [\text{x[0]}, \text{x[1]}, \ldots, \text{x[N} - 1], 0, \ldots, 0]$$

of length 2 N, zero padded with N zeros, and call $\text{X}_0 \text{[k]} = \text{DFT} \{\text{x}_0\text{[n]}\}$, for
$\text{k} = 0, \ldots, 2 \, \text{N} - 1$. Then if the sequence x[n] is real it is easy to see that

$$\sum_{k=0}^{N-1} \text{x[n] cos} \left(\frac{\pi k \, (2 \, n+1)}{2 \, N} \right) = \frac{1}{2} \, \text{Re} \, \{e^{j \, \frac{k\pi}{2N}} \, \text{X}_0 \text{[k]}\}, \quad \text{k} = 0, \ldots, \text{N} - 1$$

and therefore

$$\text{X}^{\text{II}} \text{[k]} = \frac{1}{\sqrt{2 \, \text{N}}} \, \text{C[k] Re} \, \{e^{j \, \frac{k\pi}{2N}} \, \text{X}_0 \text{[k]}\}, \quad \text{k} = 0, \ldots, \text{N} - 1.$$

■ Problem 3.29

Problem

In MATLAB generate the sequence $x[n] = n - 64$ for $n = 0, \ldots, 127$.

a) Let $X[k] = \text{DFT} \{x[n]\}$. For various values of L set to zero the "high frequency coefficients"
$X[64 - L/2] = \ldots = X[64 + L/2] = 0$ and take the IDFT. Plot the results;

b) Let $X_{\text{DCT}}[k] = \text{DCT} \{x[n]\}$. For the same values of L set to zero the "high frequency coefficients"
$X_{\text{DCT}}[127 - L] = \ldots = X[127] = 0$. Take the IDCT for each case, and compare the reconstruction with
the previous case. Comment on the error.

Solution

a) Take $L = 20, 30, 40$ and set L coefficients of the DFT to zero, as
$X[64 - L/2] = \ldots = X[64 + L/2] = 0$. Call $\hat{X}_L[k]$ the DFT we obtain in this way, and
$\hat{x}_L[n] = \text{IDFT} \{\hat{X}_L[k]\}$. Notice that $\hat{x}_L[n]$ is still a real signal, since its DFT is still symmetric around
the middle point.

Plots of $\hat{x}_L[n]$ for $L = 20, 30, 40$ show that $\hat{x}_L[n]$ is not a good approximation of $x[n]$. As expected the
errors are concentrated at the edges of the signal.
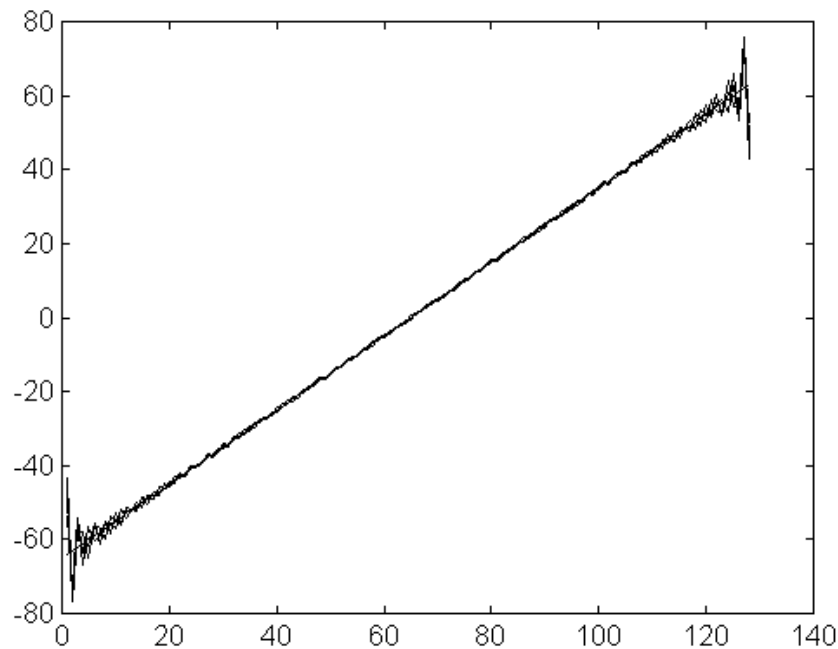
b) Now let $X_{DCT}[k] = \text{DCT}\{x[n]\}$ and set L coefficients to zero, for $L = 20, 30, 40$, as $X_{DCT}[127 - L] = ... = X_{DCT}[127] = 0$. Notice that in this case a real signal always yields a real DCT and viceversa. This is due to the basis functions being "cosines" and not "complex exponentials" as in the DFT. As a consequence we do not have to worry about symmetry to keep the signal real, as we did for the DFT in a). The inverse DCT obtained for $L = 20, 30, 40$ are shown below. Notice that the error is very small compared to the signal itself, and it is not concentrated at any particular point.
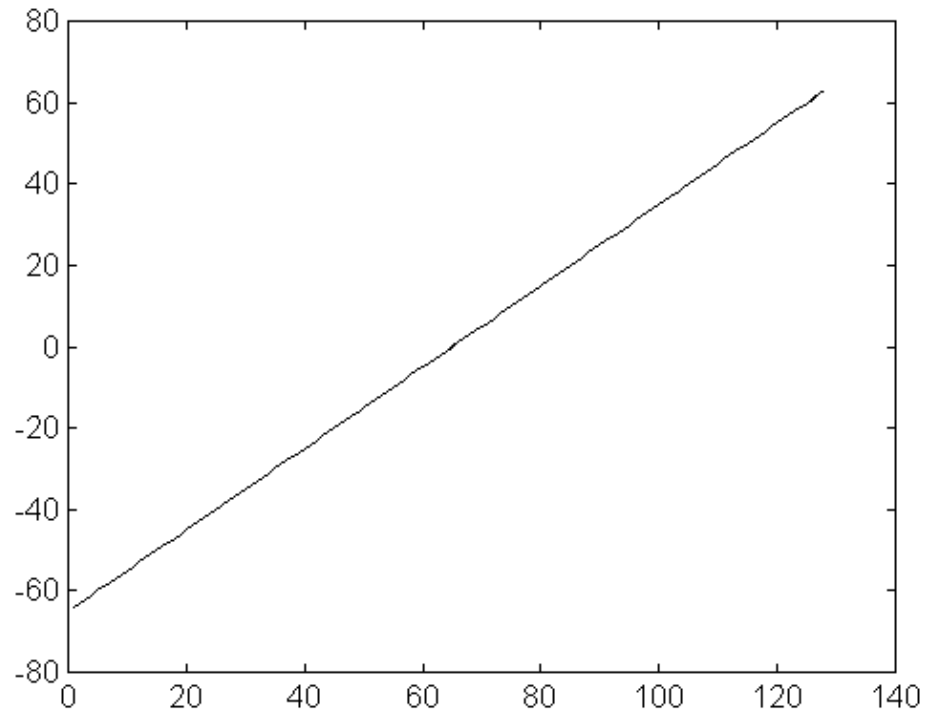
The DFT and Linear Algebra

■ Problem 3.30

Problem.

An $N \times N$ circulant matrix A is of the form

$$A = \begin{pmatrix} a[0] & a[1] & \dots & a[N-1] \\ a[N-1] & a[0] & \dots & a[N-2] \\ \vdots & \ddots & \ddots & \vdots \\ a[1] & \dots & a[N-1] & a[0] \end{pmatrix}$$

Then, do the following:

Solution.

a) Verify that $A[k, n] = a[(n - k)_N]$, for $n, k = 0, \ldots, N - 1$.

Easy, by inspection. Just notice that the first row ($k = 0$) is the sequence $a[0], \ldots, a[N - 1]$, and the $k - $th row is the first row circularly shifted by k.

b) Verify that the eigenvectors are always given by $e_k = \left[1, w_N^k, w_N^{2k}, \ldots, w_N^{(N-1)k}\right]^T$, with $k = 0, \ldots, N - 1$ and $w_N = e^{-j2\pi/N}$

By multiplying the matrix A by each vector e_k we obtain the $k -$th component of the DFT of each row. Since the $k -$th row is obtained by circularly shifting the first row k times, the DFT of the $k -$th row is DFT $\{a[(n - k)_N]\} = w_N^k X[k]$, where $X[k] = $ DFT $\{a[n]\}$ is the DFT of the first row. Therefore

$$A\, e_k = \begin{pmatrix} X[k] \\ w_N^k\, X[k] \\ \vdots \\ w_N^{(N-1)k}\, X[k] \end{pmatrix} = X[k]\, e_k$$

which shows that e_k is an eigenvector, and $X[k]$ is the corresponding eigenvalue.

c) Determine a factorization $A = E \wedge E^{*T}$ with $\wedge$ diagonal and $E^{*T} E = I$.

"Pack" all eigenvectors $e_0, \ldots, e_{N-1}$ in an $N \times N$ matrix E as

$$\overline{E} = [e_0, \ e_1, \ \ldots, \ e_{N-1}]$$

First notice that $\overline{E}^{*T} \overline{E} = N$ since

$$e_k^{*T}\, e_m = \begin{cases} 0 & \text{if } k \neq m \\ N & \text{if } k = m \end{cases}$$

Then the matrix

$$E = \frac{1}{\sqrt{N}}\, \overline{E} = \frac{1}{\sqrt{N}}[e_0, \ e_1, \ \ldots, \ e_{N-1}]$$

is such that $E^{*T} E = I$.

Then use the fact that the vectors e_k are eigenvectors, to obtain

$$A\, E = A\, \frac{1}{\sqrt{N}}[e_0, \ e_1, \ \ldots, \ e_{N-1}] = \frac{1}{\sqrt{N}}[e_0, \ e_1, \ \ldots, \ e_{N-1}] \begin{pmatrix} X[0] & 0 & \ldots & 0 \\ 0 & X[1] & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & X[N-1] \end{pmatrix}$$

which can be written as

$$A\, E = E \wedge$$

with $\wedge = \text{diag}(X[0], \ldots, X[N - 1])$. Since $E^{-1} = E^{*T}$ we obtain the desired decomposition

$$A = E \Lambda E^{*T}$$

d) Let $h[n]$ and $x[n]$ be two sequences of equal length N, and $y[n] = h[n] \otimes x[n]$ be the circular convolution between the two sequences. Show that you can write the vector $y = [y[0], ..., y[N-1]]^T$ in terms of the product $y = H x$ where $x = [x[0], ..., x[N-1]]^T$ and H circulant. Show how to determine the matrix H.

From the definition of circular convolution

$$y[n] = \sum_{k=0}^{N-1} h[(n-k)_N] \, x[k] = \sum_{k=0}^{N-1} H[k, n] \, x[k]$$

Therefore

$$y = H^T x$$

with H being circulant.

e) By writing the DFT in matrix form, and using the factorization of the matrix H, show that $Y[k] = H[k] \, X[k]$, for $k = 0, ..., N-1$, with X, H, Y being the DFT's of x, h, y respectively.

Decomposing the circulant matrix we obtain $H^T = (E \Lambda E^{*T})^T = (E^* \Lambda E^T)$ and therefore

$$y = E^* \Lambda E^T x$$

Multiplying on the left by E^T we obtain

$$E^T y = \Lambda E^T x$$

where $\Lambda = [H[0], ..., H[N-1]]$. Now

$$E^T y = \frac{1}{\sqrt{N}} \begin{pmatrix} e_0^T \\ e_1^T \\ \vdots \\ e_{N-1}^T \end{pmatrix} y = \frac{1}{\sqrt{N}} \begin{pmatrix} Y[0] \\ Y[1] \\ \vdots \\ Y[N-1] \end{pmatrix}, \quad \text{and} \quad E^T x = \frac{1}{\sqrt{N}} \begin{pmatrix} e_0^T \\ e_1^T \\ \vdots \\ e_{N-1}^T \end{pmatrix} x = \frac{1}{\sqrt{N}} \begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{pmatrix}$$

where $X[k] = \text{DFT}\{x[n]\}$ and $Y[k] = \text{DFT}\{y[n]\}$. Therefore, since $\Lambda = \text{diag}(H[0], ..., H\{N-1\})$ we obtain

$$Y[k] = H[k] \, X[k], \quad \text{for } k = 0, ..., N-1$$