# On Machin's formula with Powers of the Golden Section

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**Abstract**

In this note, we find all solutions of the equation $\frac{\pi}{4} = a\arctan(\phi^\kappa) + b\arctan(\phi^\ell)$, in integers $\kappa$ and $\ell$ and rational numbers $a$ and $b$, where $\phi$ is the golden section.

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Motivated by Machin's formula (see [Borwein and Bailey 03, p. 105])

$$\frac{\pi}{4} = 4\arctan(5^{-1}) - \arctan(239^{-1}),$$

several researchers (see, for instance, [Séroul 00] and the references therein) generalized it to identities of the form

$$\frac{k\pi}{4} = m\arctan(u^{-1}) + n\arctan(v^{-1}),$$

where $u,\ v,\ k$ are positive integers and $m, n$ are nonnegative integers. Such formulae are useful in the computation of $\pi$. It is completely natural to ask whether such formulas will hold if $u,\ v$ are replaced by numbers from a larger class than reciprocals of integers (like rational, algebraic, etc.). Chan, and Chan and Ebbing (see [Chan 07, Chan and Ebbing 06]) investigated analogues of Machin's formula with some rational coefficients when the rational numbers $5^{-1}$ and $239^{-1}$ are replaced by small powers of negative exponent of the golden section $\phi = (1+\sqrt{5})/2$ and they found three such formulas, where the pairs of exponents are $(\kappa, \ell) = (3,1), (5,3), (6,2)$. In this short note, we show that up to some trivial transformations, there are no others besides the previous ones.

**Theorem 1.** *If $\kappa, \ell \in \mathbb{Z}\backslash\{0\}$ with $|\kappa| \geq |\ell|$, $\kappa + \ell \neq 0$ and $a, b \in \mathbb{Q}$ such that*

$$\frac{\pi}{4} = a \arctan(\phi^{\kappa}) + b \arctan(\phi^{\ell}), \tag{1}$$

*then*

$$(a, b, \kappa, \ell) \in \left\{ \begin{array}{l} \left(\frac{1}{3}, \frac{1}{3}, 3, 1\right), (1, 1, -3, -1), (-1, 1, -3, 1), (1, -1, 3, -1), \\ \left(\frac{1}{5}, \frac{2}{5}, 6, 2\right), (1, 2, -6, -2), \left(-\frac{1}{3}, \frac{2}{3}, -6, 2\right), (1, -2, 6, -2), \\ \left(\frac{1}{7}, \frac{3}{7}, 5, 3\right), (1, 3, -5, -3), \left(-\frac{1}{5}, \frac{3}{5}, -5, 3\right), (1, -3, 5, -3) \end{array} \right\}.$$

Before proving the theorem, we start with a few comments about the proof, which relies on an identity of [Borwein and Borwein 87] and known facts on algebraic numbers (see [Washington 97]). When $|\kappa| > |\ell|$, we embed identity (1) in the biquadratic class number 1 field $\mathbb{K} = \mathbb{Q}[\phi, i]$, and use some known results on the prime factors of Fibonacci and Lucas sequences, to show nonexistence of solutions of (1) besides the mentioned ones. If $\kappa = \ell$, we show that the mentioned identity is equivalent to an equation in $\mathbb{Q}[\zeta_{20}]$, which has no solutions. We exclude the cases $\kappa + \ell = 0$ since it is well-known that if $x$ is a positive real number, then

$$\frac{\pi}{2} = \arctan x + \arctan\left(\frac{1}{x}\right). \tag{2}$$

We also note that whenever $(a, b, \kappa, \ell)$ is a solution of equation (1) with $a \neq 1/2$, then using the fact that $\arctan(\phi^{-\kappa}) + \arctan(\phi^{\kappa}) = \pi/2$, one gets that $(-a/(1-2a), b/(1-2a), -\kappa, \ell)$ is also a solution of equation (1).

We mention that Machin-like formulas with powers of other irrationals exist in the literature, an example being

$$\frac{\pi}{2} = 2\arctan\left(\frac{1}{\sqrt{2}}\right) + \arctan\left(\frac{1}{\sqrt{8}}\right).$$

*Proof.* Let $a = u/w,\ b = v/w$, where $u, v, w > 0$ are integers with $w$ and $\gcd(u, v)$ coprime. Then the given relation becomes

$$\frac{w\pi}{4} = u \arctan(\phi^{\kappa}) + v \arctan(\phi^{\ell}). \tag{3}$$

By the result on page 345 in [Borwein and Borwein 87], the above relation holds if and only if

$$(1-i)^w(\phi^{-\kappa}+i)^u(\phi^{-\ell}+i)^v$$

is real, which is equivalent to the fact that

$$(1-i)^w(1+i\phi^{\kappa})^u(1+i\phi^{\ell})^v = (1+i)^w(1-i\phi^{\kappa})^u(1-i\phi^{\ell})^v.$$

Raising the above equation to the fourth power and using the fact that $(1+i)^4 = (1-i)^4 = -4$, we get that

$$(1+i\phi^{\kappa})^{4u}(1+i\phi^{\ell})^{4v} = (1-i\phi^{\kappa})^{4u}(1-i\phi^{\ell})^{4v}. \tag{4}$$

If $\kappa = \ell$, we then get that

$$\left(\frac{1+i\phi^{\kappa}}{1-i\phi^{\kappa}}\right)^{4(u+v)} = 1,$$

which implies either that $u + v = 0$ or that $\zeta = (1+i\phi^{\kappa})/(1-i\phi^{\kappa})$ is a root of 1. If $u + v = 0$, then relation (3) leads to $w\pi/4 = 0$, which is impossible. Thus, $\zeta$ is a root of unity. For a positive integer $m$ we write $\zeta_m = \exp(2\pi i/m)$. Since $\phi \in \mathbb{Q}[\zeta_5]$ and $i \in \mathbb{Q}[\zeta_4]$, we get that $\zeta \in \mathbb{Q}[\phi, i] \subseteq \mathbb{Q}[\zeta_5, \zeta_4] = \mathbb{Q}[\zeta_{20}]$. Hence, $\zeta = \zeta_{20}^n$ for some $n \in \{0, \ldots, 19\}$. This implies that

$$\phi^{\kappa} = -i\left(\frac{\zeta_{20}^n - 1}{\zeta_{20}^n + 1}\right) = -\tan\left(\frac{n\pi}{20}\right).$$

One can now check that the above equation has no solution with $n \in \{0, \ldots, 19\}$ and $\kappa \in \mathbb{Z}\backslash\{0\}$.

Assume now that $|\kappa| > |\ell|$. Let $\mathbb{K} = \mathbb{Q}[\phi, i]$. It is known that $\mathbb{K}$ is biquadratic and has class number 1 (see [Hideo 86]). We shall show that $|\kappa| \le 12$. Let $\pi$ be any prime ideal dividing $1 + i\phi^{\kappa}$. From relation (4), we get that $\pi$ either divides $1 - i\phi^{\kappa}$, or it divides $1 - i\phi^{\ell}$. If $\pi$ divides $1 - i\phi^{\kappa}$, it follows that $\pi$ divides 2. Hence, if $\pi$ does not divide 2, then $\pi$ divides both $1 + i\phi^{\kappa}$ and $1 - i\phi^{\ell}$. Note that

$$N_{\mathbb{K}}(1+i\phi^{\kappa}) = (1+\phi^{2\kappa})(1+(-\phi)^{-2\kappa}) = (\phi^{\kappa}+\phi^{-\kappa})^2.$$

Hence, $N_{\mathbb{K}}(1+i\phi^\kappa)$ is either $5F_\kappa^2$ or $L_\kappa^2$ according to whether $\kappa$ is odd or even. Here, $F_m$ and $L_m$ are the regular Fibonacci and Lucas numbers given by $F_0=0,\ F_1=1,\ L_0=2,\ L_1=1$ and $F_{m+2}=F_{m+1}+F_m,\ L_{m+2}=L_{m+1}+L_m$ for all $m\geq 0$. Similarly, $N_{\mathbb{K}}(1+i\phi^\ell)$ is either $5F_\ell^2$ or $L_\ell^2$ according to whether $\ell$ is odd or even.

Assume now that $|\kappa|\geq 13$. First, assume that $\kappa$ is odd. Then there exists a prime number $p$ dividing $F_\kappa$ which is primitive; i.e., such that $p$ does not divide $F_\mu$ for any positive integer $\mu<\kappa$ (see [Bilu et al. 01] for more on primitive divisors). Let $\pi$ be any prime ideal in $\mathbb{K}$ dividing $p$. If $\pi$ divides $1+i\phi^\kappa$, then $\pi$ divides either 2 (which is impossible because $p>2$), or $\pi$ divides $1-i\phi^\ell$, which divides either $5F_\ell$ or $L_\ell$ according to whether $\ell$ is odd or even. Hence, $p$ divides either $5F_\ell$ or $L_\ell$. Since $p$ is primitive for $F_\kappa$, $p$ cannot divide $5F_\ell$. If $p\mid L_\ell$, then since $L_\ell\mid F_{2\ell}$, we then get that $\kappa\mid 2\ell$, and since $\kappa$ is odd we get that $\kappa\mid\ell$, which is impossible. It remains to show that we can always assume that $p$ is divisible by some prime in $\mathbb{K}$ dividing $1+i\phi^\kappa$. Indeed, let $\pi$ be some prime divisor of $p$. If $\pi\mid 1+i\phi^\kappa$, then we are done. If $\pi\mid 1-i\phi^\kappa$, then the complex conjugate of $\pi$ (which also divides $p$) must divide $1+i\phi^\kappa$. Finally, if $\pi$ divides $1\pm i(-\phi)^{-\kappa}$, then the image of $\pi$ via the Galois automorphism of $\mathbb{K}$ which sends $-\phi^{-1}$ to $\phi$ and fixes $i$ will send $\pi$ into a prime ideal (still dividing $p$) divisor of $1\pm i\phi^\kappa$, which is a situation already treated. This takes care of the proof of the fact that $|\kappa|\leq 12$ if $\kappa$ is odd. If $\kappa$ is even, then the same argument using the existence of primitive divisors for the Lucas sequence shows that $|\kappa|\leq 12$ also. It remains to compute the examples. Since the remaining of our analysis is based on the arithmetic structure of $F_m$ and $L_m$ for $m=\kappa,\ \ell$ and since $F_{-m}=\pm F_m$ and $L_{-m}=L_m$, we assume that $0<\ell<\kappa$ (or, we replace $\kappa$ and $\ell$ by their absolute values).

Let $\pi\nmid 2$ be a prime ideal in $\mathbb{K}$ such that

$$\pi\mid 1+i\phi^\kappa \qquad \text{and} \qquad \pi\mid 1-i\phi^\ell. \tag{5}$$

Clearly, $\pi$ cannot divide any power of $\phi$ because $\phi$ is a unit. Further, $\pi\mid(1+i\phi^\kappa)-(1+i\phi^\ell)=i\phi^\ell(1+\phi^{\kappa-\ell})$, and since $\pi\nmid i\phi^\ell$, we get that

$$\pi\mid 1+\phi^{\kappa-\ell} \qquad \text{and} \qquad \pi\mid 1-i\phi^\ell. \tag{6}$$

Furthermore, $\pi\mid\phi^{\kappa-\ell}+i\phi^\ell=i\phi^\ell(1-i\phi^{\kappa-2\ell})$, which implies that

$$\pi\mid 1-i\phi^{\kappa-2\ell} \qquad \text{and} \qquad \pi\mid 1-i\phi^\ell. \tag{7}$$

Continuing in this manner, we obtain that $\pi \mid -i\phi^{\kappa-2\ell}+i\phi^{\ell} = i\phi^{\ell}(1-\phi^{\kappa-3\ell})$, which implies that

$$\pi \mid 1-\phi^{\kappa-3\ell} \qquad \text{and} \qquad \pi \mid 1-i\phi^{\ell}. \tag{8}$$

Now let $\kappa = r\ell + t$, where $0 \le t < \ell$. Fix $\kappa \le 12$, and let $\ell \neq 0$ to be the least positive integer satisfying (5). The previous analysis suggests considering the following cases.

**Case 1.** $r \equiv 0 \pmod 4$.

From (5) and the previous analysis, we get that

$$\pi \mid 1+i\phi^{t} \qquad \text{and} \qquad \pi \mid 1-i\phi^{\ell},$$

where $t < \ell$, which implies that $\pi \mid i\phi^{t}(1+\phi^{\ell-t})$. Thus, $\pi \mid 1+\phi^{\ell-t}$ and $\pi \mid 1+i\phi^{t}$, which in turn leads to

$$\pi \mid \phi^{\ell-t} - i\phi^{t}.$$

If on the one hand $t < \ell \le 2t < 2\ell$, then $\pi \mid \phi^{\ell-t}(1-i\phi^{2t-\ell})$; that is, $\pi \mid 1-i\phi^{2t-\ell}$ with $2t-\ell < \ell$, which is in contradiction with the assumed minimality of $\ell$. If on the other hand $2t \le \ell$, we then get $\pi \mid -i\phi^{t}(1-i\phi^{\ell-2t})$. Thus, $\pi \mid 1-i\phi^{\ell-2t}$, which again either contradicts the minimality of $\ell$ (if $2t < \ell$), or the fact that $\pi \nmid 2$ (if $2t = \ell$).

**Case 2.** $r \equiv 1 \pmod 4$.

From (6), we get that

$$\pi \mid 1+\phi^{t} \qquad \text{and} \qquad \pi \mid 1-i\phi^{\ell},$$

which implies $\pi \mid \phi^{t}(1+i\phi^{\ell-t})$. So,

$$\pi \mid 1+i\phi^{\ell-t} \qquad \text{and} \qquad \pi \mid 1+\phi^{t}.$$

If $\ell \ge 2t$, then from the previous relations we get that $\pi \mid 1-i\phi^{\ell-2t}$, which is a contradiction. If $\ell < 2t$, then $\pi \mid i\phi^{\ell-t} - \phi^{t} = i\phi^{\ell-t}(1+i\phi^{2t-\ell})$, and so, $\pi \mid 1+i\phi^{2t-\ell}$. Then, since $1 \le \ell < \kappa \le 12$, we have $r \in \{1,5,9\}$ and a simple computation reveals the possibilities

$$(\kappa,\ell) \in \{(5,3),(7,4),(8,5),(9,5),(10,6),(11,6),(11,7),(12,7)\}. \tag{9}$$

**Case 3.** $r \equiv 2 \pmod 4$.

From (7), we get that

$$\pi \mid 1 - i\phi^t \qquad \text{and} \qquad \pi \mid 1 - i\phi^\ell$$

with $t < \ell$, which contradicts again the minimality of $\ell$.

**Case 4.** $r \equiv 3 \pmod 4$.

From (8), we get

$$\pi \mid 1 - \phi^t \qquad \text{and} \qquad \pi \mid 1 - i\phi^\ell.$$

If on the one hand $t \neq 0$, then $\pi \mid -\phi^t + i\phi^\ell = -\phi^t(1 - i\phi^{\ell-t})$, which implies $\pi \mid 1 - i\phi^{\ell-t}$, in contradiction with the minimality of $\ell$. If on the other hand $t = 0$, then $\kappa = r\ell$, with $r \in \{3, 7\}$, and we obtain the possibilities

$$(\kappa, \ell) \in \{(3,1), (6,2), (7,1), (9,3), (12,4)\}. \tag{10}$$

The authors of [Chan and Ebbing 06] found the Machin-like formulas with powers $(\kappa, \ell)$ of the golden section, where $(\kappa, \ell) = (3,1), (5,3), (6,2)$; that is,

$$\begin{aligned}
\frac{3\pi}{4} &= \arctan(\phi) + \arctan(\phi^3), \\
\frac{5\pi}{4} &= 2\arctan(\phi^2) + \arctan(\phi^6), \\
\frac{7\pi}{4} &= 3\arctan(\phi^3) + \arctan(\phi^5).
\end{aligned}$$

To get the remaining examples listed in the statement of Theorem 1, we note that whenever $(a, b, \kappa, \ell)$ is a solution of equation (1) with $a \neq 1/2$, then using the fact that $\arctan(\phi^{-\kappa}) + \arctan(\phi^\kappa) = \pi/2$, one gets that $(-a/(1-2a), b/(1-2a), -\kappa, \ell)$ is also a solution of equation (1). The remaining nine solutions of equation (1) are all obtained in the above fashion from the above three solutions with $(\kappa, \ell) = (3,1), (5,3), (6,2)$.

We now need to deal with the other pairs in (9) and (10), namely

$$\begin{aligned}(\kappa, \ell) \in \{&(7,1), (7,4), (8,5), (9,3), (9,5), \\ &(10,6), (11,6), (11,7), (12,4), (12,7)\}.\end{aligned} \tag{11}$$

Assume, say that $(\kappa, \ell) = (7, 1)$. Take a prime ideal $\pi$ in $\mathbb{K}$ such that $\pi$ divides both $1 + i\phi^7$ and 13 (note that $F_7 = 13$). Such a prime ideal divides neither 2, nor $1 - i\phi$, since otherwise the rational prime 13 would divide $N_{\mathbb{K}}(1 - i\phi) = (\phi + \phi^{-1})^2 = 5$, and so we get a contradiction. Similarly, we can remove all the remaining possibilities from (11) since in each instance there is a rational prime divisor of $5F_\kappa^2$ (with $\kappa$ odd), or $L_\kappa^2$ (for $\kappa$ even) which is not a rational prime divisor of the corresponding norm of $1 - i\phi^\ell$.

We are not yet done, since so far we have merely shown that if $(a, b, \kappa, \ell)$ with $|\kappa| \geq |\ell|$ and $\kappa + \ell \neq 0$ satisfies equation (1), then $(|\kappa|, |\ell|) = (3, 1), (5, 3), (6, 2)$. In order to finish, we need to show that each such solution $(a, b, \kappa, \ell)$ is uniquely determined by its last two components. Assume that this is not so. Then there exists a pair $(\kappa, \ell)$ such that $(a, b, \kappa, \ell)$ and $(a', b', \kappa, \ell)$ are both solutions of equation (1) for two distinct pairs $(a, b)$ and $(a', b')$ of rational numbers. It then follows that $\arctan(\phi^\ell)$ and $\pi$ are linearly dependent over the rationals. Thus, there exists a rational number $r$ such that $\phi^\ell = \tan(r\pi)$. By replacing $r$ with $1/2 - r$, we may assume that $\ell > 0$. Then $(2\cos(r\pi))^2 = 4/(1 + \phi^{2\ell})$. However, $2\cos(r\pi) = e^{ir\pi} + e^{-ir\pi}$ is an algebraic integer. Thus, $4/(1 + \phi^{2\ell})$ is an algebraic integer. When $\ell = 1$ and 2 this last number takes the values

$$\frac{2(\sqrt{5} - 1)}{\sqrt{5}} \qquad \text{and} \qquad \frac{2(3 - \sqrt{5})}{3},$$

and none of them is an algebraic integer. Since $\dfrac{4}{1 + \phi^2} = (1 - \phi^2 + \phi^4) \cdot \dfrac{4}{1 + \phi^6}$, we get that if the number $4/(1 + \phi^{2\ell})$ is an algebraic integer when $\ell = 3$, then it is also for $\ell = 1$, and we have just seen that this is impossible. This indeed completes the proof of Theorem 1. $\square$

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