Fibonacci numbers that are not sums of two prime powers

Florian Luca\textsuperscript{1} and Pantelimon Stănică\textsuperscript{2}

\textsuperscript{1} IMATE, UNAM, Ap. Postal 61-3 (Xangari), CP. 58 089
Morelia, Michoacán, Mexico; e-mail: fluca@matmor.unam.mx

\textsuperscript{2} Auburn University Montgomery, Department of Mathematics,
Montgomery, AL 36124-4023, USA; e-mail: pstanica@mail.aum.edu

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Abstract

In this paper, we construct an infinite arithmetic progression $A$ of positive integers $n$ such that if $n \in A$, then the $n$th Fibonacci number is not a sum of two prime powers.

1 Introduction

In 1849, A. de Polignac [7] asked if every odd positive integer can be represented as the sum of a power of 2 and a prime (or 1). Euler did note that 959 was not of this form. Romanoff [8] used the Brun sieve to show that a positive proportion of integers are representable in this way. Later, van der Corput [2] showed that a positive proportion of integers are not representable in this way by using covering congruences. With the same method as van der Corput’s, Erdős [3] constructed a residue class of odd numbers which contains no integers of the above form. Extending Erdős’s argument, Cohen and Selfridge [1] constructed a 26 digit number which is neither the sum nor the difference of two prime powers. Inspired by their work, Z.W. Sun (see [9]) constructed a residue class of odd integers consisting exclusively of numbers not of the form $\pm p^a \pm q^b$ with some primes $p$ and $q$ and some nonnegative integers $a$ and $b$. We mention that Erdős asked if there exist infinitely many positive integers which are not representable as a sum or difference of two powers (see [5]) and a partial result can be found in [6].

In this paper, we show that there exist infinitely many positive integers which are not of the form $p^a + q^b$ with primes $p$ and $q$ and nonnegative integers $a$ and $b$ and which further can be chosen to be members of the Fibonacci sequence $(F_n)_{n \geq 0}$ given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$.

In what follows, we use the Vinogradov symbols $\gg$ and $\ll$ with their usual meanings. We recall that given two functions $A$ and $B$ of the real variable $x$, the notations $A \ll B$ and $B \gg A$ are equivalent to the fact that the inequality $|A(x)| \leq cB(x)$ holds with some positive constant $c$ and for all sufficiently large real numbers $x$.

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2 The Result

Our main result is the following.

Theorem 1. There exists a positive integer \( n_0 \) such that if \( n > n_0 \) and

\[
    n \equiv 1807873 \pmod{3543120}
\]

then \( F_n \neq p^a + q^b \) with \( p, q \) prime numbers and \( a, b \) nonnegative integers.

We use the same method employed in \([2, 3, 9]\). However, there are additional difficulties which arise because we want our numbers to belong to the Fibonacci sequence.

Let us quickly recall how one can create a residue class of odd integers most of which are not sums of two powers. Well, assume that \( \text{gcd}\left(2^a + q^b\right) = 1 \). We write \( 2^a + q^b = p \). Hence, \( p \) divides \( 2^a + q^b - 1 \) but not \( 2^a \) or \( q^b \).

We now modify the above construction in order to insure that our numbers can be chosen from the Fibonacci sequence. Let \( k \) be a positive integer. It is known that \((F_n)_{n \geq 0}\) is periodic modulo \( k \). We write \( h(k) \) to denote this period. Moreover, for integers \( f \) and \( k \) we write \( \mathcal{A}(f, k) \) for the set of residue classes \( n \) modulo \( h(k) \) such that \( F_n \equiv f \pmod{k} \).

Assume now that \((a_i, b_i, p_i)_{i=1}^s\) is a finite set of triples of nonnegative integers \( a_i \) and \( b_i \) and distinct odd primes \( p_i \) for \( i = 1, \ldots, s \) which fulfill the following conditions:

(i) For every \( a \in \mathbb{Z} \) there exists \( i \in \{1, \ldots, s\} \) such that \( a \equiv a_i \pmod{b_i} \).

(ii) \( p_i \mid (2^{b_i} - 1) \) holds for all \( i = 1, \ldots, s \).

(iii) The set

\[
    \bigcap_{i=1}^{s} \mathcal{A}(2^{a_i}, p_i) \neq \emptyset.
\]

Moreover, if there exists \( i \in \{1, \ldots, s\} \) such that \( 3 \mid h(p_i) \), then we shall assume that the above intersection contains a class coprime to 3.
Let $x$ be an element of the set $\cap_{s=1}^{n} A(2^{a_i}, p_i)$. Note that $x$ is defined only modulo \( M = \text{lcm}[h(p_1), \ldots, h(p_s)] \). Moreover, if $3 \mid M$, then $x$ is not a multiple of 3. If $M$ is not a multiple of 3 we replace $M$ by $3M$ and $x$ by the solution of the system of congruences $x \equiv x \pmod{3}$. Assume now that $n \equiv x \pmod{M}$ and that $F_n = p^a + q^b$ holds with some primes $p$ and $q$ and nonnegative integers $a$ and $b$. It then follows that $F_n$ is an odd integer, because the only even Fibonacci numbers are those whose indices are multiples of 3. Since $F_n$ is odd, it follows that one of $p$ and $q$, say $p$ is 2. By (i) above, there exists $i \in \{1, \ldots, s\}$ such that $a \equiv a_i \pmod{b_i}$. By (ii) above, it follows that $2^a \equiv 2^{a_i} \pmod{p_i}$. By the choice of $n$, we have that $F_n \equiv 2^{2r} \pmod{p_i}$. Thus, $F_n \equiv 2^a \pmod{p_i}$. In particular, $p_i \mid (F_n - 2^a)$. However, since $F_n - 2^a = q^b$, it follows that $q = p_i$. Thus, $q \in \{p_1, \ldots, p_s\}$. We now get that $F_n = p^a + q^b$ for some $s = 1, \ldots, s$. Since $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, we may apply a well-known result from the theory of $S$-unit equations (see [4]) to conclude that such an equation can have only finitely many solutions $(n, a, b)$. Thus, there exists $n_0$ with the property that if $n > n_0$ and $n \equiv x \pmod{M}$, then $F_n$ is not of the form $p^a + q^b$ with primes $p$ and $q$ and nonnegative integers $a$ and $b$.

In order to finish the proof of the Theorem, it suffices to find a finite set of triples $(a_i, b_i, p_i)_{i=1}^{s}$ fulfilling (i)-(iii) above.

We first note that by taking $s = 7$ and

\[((a_1, b_1), \ldots, (a_7, b_7)) = ((0, 2), (0, 3), (3, 4), (1, 12), (5, 36), (17, 36), (29, 36))\]

we get that (i) above is fulfilled. Indeed, every integer is congruent either to 0 (mod 2) or to 0 (mod 3) or to 3 (mod 4) or to 1 (mod 12) or to 5 (mod 12), and in this last case it is congruent to either 5, 17 or 29 modulo 36. We now take $(p_1, \ldots, p_7) = (3, 7, 5, 13, 19, 37, 73)$ and note that condition (ii) above is fulfilled. It is easy to check that $(h(p_1), \ldots, h(p_7)) = (8, 16, 20, 28, 18, 76, 148)$. Finally, it is easy to see that

\[
A(2^{a_1}, p_1) = A(2^0, 3) = \{1, 2, 7\} \pmod{8},
\]
\[
A(2^{a_2}, p_2) = A(2^0, 7) = \{1, 2, 6, 15\} \pmod{16},
\]
\[
A(2^{a_3}, p_3) = A(2^3, 5) = \{4, 6, 7, 13\} \pmod{20},
\]
\[
A(2^{a_4}, p_4) = A(2^1, 13) = \{3, 25\} \pmod{28},
\]
\[
A(2^{a_5}, p_5) = A(2^5, 19) = \{7, 11\} \pmod{18},
\]
\[
A(2^{a_6}, p_6) = A(2^{17}, 37) = \{10, 15, 28, 61\} \pmod{76},
\]
\[
A(2^{a_7}, p_7) = A(2^{29}, 73) = \{53, 95\} \pmod{148},
\]

One now checks that the system of congruences $x \equiv 1 \pmod{8}$, $x \equiv 1 \pmod{16}$, $x \equiv 13 \pmod{20}$, $x \equiv 25 \pmod{28}$, $x \equiv 7 \pmod{18}$, $x \equiv 61 \pmod{76}$ and $x \equiv 53 \pmod{148}$ has a positive integer solution $x$. In fact, the above system is equivalent to the system of congruences $x \equiv 1 \pmod{16}$, $x \equiv 3 \pmod{5}$, $x \equiv 4 \pmod{7}$, $x \equiv 7 \pmod{9}$, $x \equiv 4 \pmod{19}$ and $x \equiv 16 \pmod{37}$. Solving this system, we get $x \equiv 1807873 \pmod{3543120}$, which, together with the above arguments completes the proof of Theorem 1.

We would like to conclude by offering the following problem.

**Problem.** Find an arithmetic progression of positive integers $A \pmod{B}$ such that if $n > n_0$ satisfies $n \equiv A \pmod{B}$, then $F_n \neq \pm p^a \pm q^b$ for any primes $p$, $q$ and nonnegative integers $a$, $b$.

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References


