Cholesky Factorizations of Matrices Associated with \(r\)-Order Recurrent Sequences

Pantelimon Stănică *

Auburn University Montgomery, Department of Mathematics
7061 Senator’s Drive, Montgomery, AL 36117
e-mail: pstanica@mail.aum.edu

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Abstract

In this paper we extend some results on the factorization of matrices associated to Lucas, Pascal, Stirling sequences by the Fibonacci matrix. We provide explicit factorizations of any matrix by the matrix associated with an \(r\)-order recurrent sequence \(U_n\) (having \(U_0 = 0\)). The Cholesky factorization for the symmetric matrix associated to \(U_n\) is also obtained.

1 Introduction and Motivation

Shapiro, et.al. [9] introduced the concept of Riordan matrix (element of the Riordan group \(\mathcal{R}\)) in an attempt to develop a tool to deal with large classes of combinatorial identities. An infinite lower triangular matrix \(L = (l_{n,k})_{n,k \geq 0}\) is a Riordan matrix if there exist generating functions \(g(z) = \sum g_n z^n, f(z) = \sum f_n z^n, f_0 = 0, f_1 \neq 0,\) such that \(l_{n,0} = g_n\) and \(\sum_{n \geq k} l_{n,k} z^n = g(z)(f(z))^k.\) We write \(L = (g(z), f(z))\) (or \((g, f)\)). Given two Riordan matrices \(L_1 = (g_1, f_1)\) and \(L_2 = (g_2, f_2)\) we define the multiplication by \(L_1L_2 = (g_1 g_2(f_1), f_2(f_1))\), and the inverse \(L^{-1} = (\frac{1}{g(f)}, \tilde{f}),\) where \(\tilde{f}\) is the compositional inverse of \(f.\) The identity matrix \(I = (1, z)\) is Riordan. Thus, \(\mathcal{R}\) becomes a group under the previous operation. The underlying idea is to find factorizations of some matrices by matrices for which one knows their Riordan components \((g, f).\) Using the operation of \(\mathcal{R},\) then one can find closed forms for some combinatorial sums. In [8] the authors proved that some matrices in the Riordan group can be factorized as \(R = PCF,\) where \(P\) is a Pascal-like matrix, \(C\) a Catalan sequence connected matrix and \(F\) a Fibonacci involving matrix.

In [1, 3, 4, 5, 6, 10] the authors concentrated on various factorizations of the Fibonacci matrix, Pascal, and Stirling matrices. They also studied the Cholesky factorization of the Fibonacci matrix. Besides the interest in its own right, Cholesky factorization can be applied in the resolution of linear equations. A linear equation \(Ax = b\) can be solved twice as fast using its Cholesky factorization \(A = LL^T,\) than by LU-factorization.

*Associated with the Institute of Mathematics “Simion Stoilow” of The Romanian Academy, Bucharest, Romania
The mentioned results were generalized in a recent paper [3] to a $k$-Fibonacci sequence. In [2] the author gave a few applications of the $k$-Lucas sequences in graph theory, and in [6] a few applications of the $k$-Fibonacci numbers in combinatorics and probabilities were provided.

It is our intent in this paper to point out that many of the results which were obtained in all these papers can be generalized to factorizations of any matrix by matrices constructed using any $r$-order recurrent sequence (having $U_0 = 0$ - see next). The Cholesky factorization for the associated symmetric matrix is also obtained here.

Since it is easier to write it down, we prove the results explicitly in the case of a second-order recurrent sequence pointing out in the general case only the differences (if they exist).

The present author was led to this investigation in an attempt to find divisors in Cayley graphs by looking at factorizations of the adjacency matrix of a graph.

2 Binary Recurrent Sequences

We consider the general nondegenerate second-order (binary) recurrent sequence $U_{n+1} = aU_n + bU_{n-1}$, $a, b, U_0, U_1$ integers, $\delta = \sqrt{a^2 + 4b} \neq 0$ real. We recall the Binet formula for the sequence $U_n$, namely $U_n = A\alpha^n - B\beta^n$, where $\alpha = \frac{1}{2}(a + \sqrt{a^2 + 4b}), \beta = \frac{1}{2}(a - \sqrt{a^2 + 4b})$ and $A = \frac{U_1 - U_0\beta}{\alpha - \beta}, B = \frac{U_1 - U_0\alpha}{\alpha - \beta}$. In this paper we assume that $U_0 = 0$, therefore $A = B$. We associate the sequence $V_n = \alpha^n + \beta^n$, which satisfies the same recurrence, with the initial conditions $V_0 = 2, V_1 = a$. Obviously, if $a = b = 1$ and $U_0 = 0, U_1 = 1$, then $U_n = F_n$, the Fibonacci sequence, and $V_n = L_n$, the Lucas sequence.

We define the Pascal matrix $P_n$ with entries $\binom{i-1}{j-1}$ (with the standard convention that the binomial coefficients are equal to 0 if $i < j$).

Let $U_n$ be the matrix with entries

$$u_{i,j} = \begin{cases} U_{i-j+1} & \text{if } i - j + 1 \geq 0 \\ 0 & \text{if } i - j + 1 < 0. \end{cases}$$

If $U_n = F_n$ the Fibonacci sequence, then $U_n = F_n$. In [5], the inverse of $F_n$ was found to be the matrix $F_n^{-1}$ with entries

$$f'_{i,j} = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i - 2 \leq j \leq i - 1 \\ 0 & \text{otherwise}. \end{cases}$$

Define $X_n(\gamma) = (x_{i,j}(\gamma))_{i,j}$, where

$$x_{i,j}(\gamma) = \begin{cases} \gamma^{i-j+1} & \text{if } i - j + 1 \geq 0 \\ 0 & \text{otherwise}. \end{cases}$$

For instance,

$$X_5(\gamma) = \begin{pmatrix} \gamma & 0 & 0 & 0 & 0 \\ \gamma^2 & \gamma & 0 & 0 & 0 \\ \gamma^3 & \gamma^2 & \gamma & 0 & 0 \\ \gamma^4 & \gamma^3 & \gamma^2 & \gamma & 0 \\ \gamma^5 & \gamma^4 & \gamma^3 & \gamma^2 & \gamma \end{pmatrix}.$$
Proposition 1. The inverse of $X_n(\gamma)$ is the matrix with entries

$$
\begin{cases}
1/\gamma & \text{if } i = j \\
-1 & \text{if } j = i - 1 \\
0 & \text{otherwise}
\end{cases}
$$

Proof. Straightforward. \qed

Proposition 2. The inverse $\mathcal{U}_n^{-1}$ of $\mathcal{U}_n$ is the matrix with entries $u'_{i,j}$, where

$$
u'_{i,j} = \begin{cases}
1 & \text{if } i = j \\
-a & \text{if } j = i - 1 \\
-b & \text{if } j = i - 2 \\
0 & \text{otherwise}
\end{cases}
$$

Proof. If $i = j$, then $\sum_{k=1}^{n} u_{i,k} u'_{k,i} = 1$. Assume $i \neq j$. Then,

$$
\sum_{k=1}^{n} u_{i,k} u'_{k,j} = \frac{1}{U_1} (u_{i,j} - au_{i,j+1} - bu_{i,j+2}) = \frac{1}{U_1} (U_{i-j+1} - aU_{i-j} - bU_{i-j+1}) = 0.
$$

The proposition is proved. \qed

Theorem 2.1 of [4] states that the Pascal matrix

$$
\mathcal{P}_n = \mathcal{S}_n \mathcal{F}_n
$$

where $\mathcal{L}_n$ is the matrix with entries $\binom{i}{j-1} - \binom{j-2}{i-1} - \binom{i-3}{j-1}$. Also, Theorem 3.1 of [4] states that

$$
\mathcal{S}_n(2) = \mathcal{F}_n \mathcal{M}_n
$$

and Theorem 4.1 of [4] states that

$$
\mathcal{S}_n(1) = \mathcal{Q}_n \mathcal{F}_n
$$

where $\mathcal{S}_n(2), \mathcal{S}_n(1)$ are the matrices corresponding to the Stirling numbers of the first and second kind and $\mathcal{M}_n, \mathcal{Q}_n$ are defined suitably using the Stirling matrices.

The previous results are not specific to the Fibonacci, Pascal, or Stirling matrices, rather they can be generalized to any matrix $T_n$. We do that in our first theorem. For an arbitrary matrix $T_n = (t_{i,j})_{1 \leq i,j \leq n}$ and a binary sequence $U = (U_k)$ define the matrix $T_n(U)$ with entries $t'_{i,j} = t_{i,j} - at_{i-1,j} - bt_{i-2,j}$ if $i \geq 3$, and $t'_{i,j} = t_{i,j}$ if $i \leq 2$. (We could give up the extra definition of $t'_{i,j}$ for $i \leq 2$, if we assume that the matrix $T_n$ is padded by zeros, so $t_{i,j} = 0$ if one of the indices is $\leq 0$.)

Theorem 3. Let $T_n(U)$ be the matrix associated to an arbitrary matrix $T_n$ and the sequence $U = (U_k)_k$, defined as above. Then the matrix $T_n$ can be factorized by $\mathcal{U}_n$ as

$$
T_n = \frac{1}{U_1} \mathcal{U}_n T_n(U).
$$

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Proof. We previously proved that the matrix $U_n$ is invertible. Thus, it suffices to show now that $T_n(U) = U_1U_n^{-1}T_n$. Since the entries of $U_n^{-1}$ are $1/U_1$ if $i = j$, $-a/U_1$ if $j = i - 1$, $-b/U_1$ if $j = i - 2$ and 0 otherwise, a straightforward computation proves our claim. 

As a simple corollary we have the following combinatorial identity.

**Corollary 4.** If $i \geq j$, then

$$t_{i,j} = \frac{1}{U_1} \sum_{k=1}^{i} U_{i-k+1}(t_{k,j} - at_{k-1,j} - bt_{k-2,j}).$$

One can obtain a plethora of seemingly complicated combinatorial identities by taking various suitably chosen matrices $T_n$ and sequences $(U_k)_k$. In particular, one can obtain all combinatorial identities of [4]. As a simple example, by taking $t_{i,j} = x^{i-j}\binom{i}{j}$ ($i \geq j$) we obtain

$$\sum_{k=j}^{i} U_{i-k+1} x^k \left( \binom{k}{j} - a \binom{k-1}{j} x^{-1} - b \binom{k-2}{j} x^{-2} \right) = U_1 x^i \binom{i}{j}.$$

Or, taking $t_{i,j} = U_{i-j+1}\binom{i}{j}$ ($i \geq j$) we get

$$\sum_{k=j}^{i} U_{i-k+1} \binom{k}{j} \left( U_{k-j+1} - aU_{k-j} \frac{k-j}{k} - bU_{k-j-1} \frac{(k-j)(k-j-1)}{k(k-1)} \right) = U_1 U_{i-j+1} \binom{i}{j}.$$

The reader can amuse oneself, by taking arbitrary entries $t_{i,j}$ and derive other combinatorial identities.

Define the direct sum of two square matrices (not necessarily of the same dimension) $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Further, we define the matrices

$$S_0 = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix}; \quad S_{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix} \quad (1)$$

$$S_k = S_0 \oplus I_k; \quad U_k = [U_1] \oplus U_{k-1}; \quad G_1 = I_n; \quad G_2 = I_{n-1} \oplus S_{-1}; \quad G_k = I_{n-k} \oplus S_{k-3}, \quad (n \text{ is fixed}); \quad (2)$$

$$C_k = \begin{pmatrix} U_1 & 0 & 0 & \cdots & 0 \\ U_2 & 1 & 0 & \cdots & 0 \\ U_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ U_k & 0 & 0 & \cdots & 1 \end{pmatrix}. \quad (4)$$

We regard $C_k$ as a companion matrix to the sequence $U = (U_n)_n$.

We prove now

**Theorem 5.** The following identities are true

$$U_k S_{k-3} = U_k, \quad k \geq 3,$$

$$U_n = U_1 G_1 G_2 \cdots G_n,$$

$$U_n = C_n (I_1 \oplus C_{n-1})(I_2 \oplus C_{n-2}) \cdots (I_{n-1} \oplus C_1).$$

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Proof. If \( k = 3 \), then we need to prove \(((U_1 \oplus U_2)S_0 = U_3\) \&\& \((U_1 \oplus U_2)S_1 = U_4\). This follows from:

\[
\begin{pmatrix}
U_1 & 0 & 0 \\
0 & U_1 & 0 \\
0 & U_2 & U_1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
a & 1 & 0 \\
b & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
U_1 & 0 & 0 \\
U_2 & U_1 & 0 \\
U_3 & U_2 & U_1
\end{pmatrix},
\]

which is certainly true. The general case follows in the same manner observing that in \(((U_1 \oplus U_{k-1}) (S_0 \oplus I_{k-3})\) multiplying the \( i \)th row of the first matrix by the \( 1 \)st row of the second matrix renders \( U_{i+1} \) (using the recurrence on \((U_k)\)) and multiplying the \( i \)th row by the \( 2 \)nd, 3rd column etc., we get \( U_i, U_{i-1}, etc. \)

We prove now the factorization of \( U_n \). By the first part of our theorem, we have \( U_n = U_nS_{n-3} = U_nG_n \). Further,

\[
U_n = [U_1] \oplus U_{n-1} = [U_1] \oplus U_{n-1}S_{n-4} = ([U_1] \oplus U_{n-1})(I_1 \oplus S_{n-4}) = ([U_1] \oplus U_{n-1})G_{n-1}.
\]

Continuing in this manner, and observing that \([U_1] \oplus [U_1] \oplus \cdots \oplus [U_1] = U_1I_n = U_1G_1\), we obtain the result. The last identity is done similarly.

The following is an example of the second identity

\[
\begin{pmatrix}
U_1 & 0 & 0 \\
0 & U_1 & 0 \\
0 & U_2 & U_1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
a & 1 & 0 \\
b & 0 & 1
\end{pmatrix}
= U_1
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
a & 1 & 0 \\
b & 0 & 1
\end{pmatrix}.
\]

One might think that Theorem 2.1 of [5] is incomplete, that is, that the product should end in \( I_{n-1} \oplus C_1 \). However, if \( U_n = F_n \), the Fibonacci sequence, then \( I_{n-1} \oplus C_1 = I_n \), which is not necessarily the case in this paper.

**Corollary 6.** By Theorem 5, we have

\[
U_n^{-1} = (I_{n-1} \oplus C_1)^{-1}(I_{n-2} \oplus C_2)^{-1} \cdots (I_1 \oplus C_{n-1})^{-1}C_n^{-1}.
\]

where

\[
C_k^{-1} = \begin{pmatrix}
U_1/U_1 & 0 & 0 & \cdots & 0 \\
-U_2/U_1 & 1 & 0 & \cdots & 0 \\
-U_3/U_1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-U_k/U_1 & 0 & 0 & \cdots & 1
\end{pmatrix}.
\]


3 Cholesky Factorization

We define now the \( n \times n \) symmetric \( U \)-matrix \( S_n(U) \) with the entries given by (for \( i \leq j \))

\[
q_{i,j} = \begin{cases}
\sum_{k=1}^{i} U_k^2 & \text{if } i = j \\
a q_{i,j-1} + b q_{i,j-2} & \text{if } i + 1 \leq j,
\end{cases}
\]

with the convention that \( q_{i,j} = 0 \) if \( j \leq 0 \).

Next we give explicit expressions for \( q_{i,j} \)'s.
Lemma 7. We have

\begin{align*}
q_{1,j} &= q_{j,1} = U_1 U_j \quad \text{and} \\
q_{i,j} &= U_i U_j + U_{i-1} U_{j-1} + \cdots + U_1 U_{j-i+1}, \quad \text{if } j \geq i \geq 2
\end{align*}

Proof. The first relation \( q_{1,j} = q_{j,1} = U_1 U_j \) follows easily from the recurrence on \( q_{i,j} \)'s. We show that \( q_{i,j} = U_i U_j + U_{i-1} U_{j-1} + \cdots + U_1 U_{j-i+1} \) by induction. The initial conditions certainly hold. Adding the relations

\begin{align*}
aq_{i,j-1} &= a_U U_{j-1} + aU_{i-1} U_{j-2} + \cdots + aU_1 U_{j-i} \\
bq_{i,j-2} &= bU_{i-2} U_{j-1} + bU_{i-1} U_{j-3} + \cdots + bU_1 U_{j-i-1}
\end{align*}

and using the recurrence \( U_{k+1} = aU_k + bU_{k-1} \) we obtain the expression of \( q_{i,j} \). \( \square \)

Let \( M^T \) be the transpose of \( M \).

Theorem 8. The Cholesky factorization of \( S_n(U) \) is given by \( S_n(U) = U_n U_n^T \).

Proof. It suffices to prove that \( U_n U_n^{-1} S_n(U) = U_n U_n^T \). Proposition 2 implies that the matrix \( U_n U_n^{-1} S_n(U) \) has the entries \( x_{i,j} \), where

\[
x_{i,j} = \begin{cases} q_{1,j} (= U_1 U_j), & \text{if } i = 1, \\ q_{2,j} - aq_{1,j}, & \text{if } i = 2, \\ q_{i,j} - aq_{i-1,j} - bq_{i-2,j} & \text{if } i \geq 3. \end{cases}
\]

Thus, we need to prove that \( x_{i,j} \) are the entries of \( U_n U_n^T \), that is \( x_{i,j} = u_{j,i} = U_1 U_{j-i+1} \). For \( i = 1, 2 \), we note that the claim follows from \( q_{1,j} = U_1 U_j \) and \( q_{2,j} - aq_{1,j} = U_2 U_j + U_1 U_{j-1} - aU_1 U_j = U_1 U_{j-1} \), since \( U_2 = aU_1 \). In general,

\[
q_{i,j} - aq_{i-1,j} - bq_{i-2,j} = U_i U_j + U_{i-1} U_{j-1} + \cdots + U_3 U_{j-2} + U_2 U_{j-1} + U_1 U_{j-i+1} - aU_{i-1} U_j - aU_{i-2} U_{j-1} - \cdots - aU_2 U_{j-i-1} - aU_1 U_{j-i} - bU_{i-2} U_{j-1} - \cdots - bU_1 U_{j-i-1}
\]

\[
= (U_i - aU_{i-1} - bU_{i-2}) U_j + \cdots + (U_3 - aU_2 - bU_1) U_{j-i-1} + (U_2 - aU_1) U_{j-i} + U_1 U_{j-i+1}
\]

using the recurrence on the sequence \( (U_k)_k \). \( \square \)

4 \( r \)-Order Recurrent Sequences

In this section we generalize our previous results. We started with the particular case since the proofs are easier to write, but the general case is treated similarly. Assume \( r \) is a fixed positive integer. Let \( U = (U_n)_n \) be an \( r \)-order recurrence, that is, \( U_n \) satisfies the recurrence

\[
U_{n+r} = a_1 U_{n+r-1} + a_2 U_{n+r-2} + \cdots + a_r U_n,
\]

and for simplicity we assume that \( U_0 = U_1 = \ldots = U_{2-r} = 0, \ U_1 \) arbitrary. Therefore, the Binet formula gives

\[
U_n = A_1 \alpha_1^n + A_2 \alpha_2^n + \cdots + A_r \alpha_r^n,
\]
for some suitable $A_1, \cdots, A_r$ (found by using Cramer’s formulas from linear algebra applied to the system obtained by imposing the initial conditions), where $\alpha_i$ are the complex roots of the characteristic $k$-degree polynomial of (7). We associate a matrix $\mathcal{U}_n$ to this sequence, with entries

$$u_{i,j} = \begin{cases} U_{i-j+1} & \text{if } i-j+1 \geq 0 \\ 0 & \text{if } i-j+1 < 0. \end{cases}$$

**Proposition 9.** The inverse of $\mathcal{U}_k$ is the matrix with entries $u'_{i,j}$, where

$$U_1 u'_{i,j} = \begin{cases} 1 & \text{if } i = j \\ -a_k & \text{if } j = i - k, \text{ where } k = 1, 2, \ldots, r \\ 0 & \text{otherwise} \end{cases}$$

**Proof.** Straightforward, as in the proof of Proposition 2. \hfill $\Box$

Now we show that any matrix $T_n = (t_{i,j})_{i,j=1,2,\ldots,n}$ can be factorized by $\mathcal{U}_n$. We define $T_n(\mathcal{U})$ with entries $t'_{i,j}$ where

$$t'_{i,j} = t_{i,j} - \sum_{k=1}^{r} a_k t_{i-k,j},$$

where we regard $T_n$ as being padded with zeros, so the previous relation makes sense for negative indices.

**Theorem 10.** We have

$$T_n = \frac{1}{U_1} \mathcal{U}_n T_n(\mathcal{U}).$$

**Proof.** The proof follows closely the one of Theorem 3. \hfill $\Box$

**Corollary 11.** If $i \geq j$, then

$$t_{i,j} = \frac{1}{U_1} \sum_{k=1}^{i} U_{i-k+1}(t_{k,j} - a_1 t_{k-1,j} - a_2 t_{k-2,j} - \cdots - a_r t_{k-r,j}).$$

5 General Cholesky Factorization

As before, we define now the (generalized) $n \times n$ symmetric matrix $\mathcal{S}_n(\mathcal{U})$ with the entries given by ($i \leq j$)

$$q_{i,j} = q_{j,i} = \begin{cases} \sum_{k=1}^{i} U_k^2 & \text{if } i = j \\ \sum_{k=1}^{i-1} a_k q_{i,j-k} & \text{if } i + 1 \leq j, \end{cases}$$

with the convention that $q_{i,j} = 0$ if $j \leq 0$. One can prove as in the case of $r = 2$ the following lemma.

**Lemma 12.** We have for $i \leq j$,

$$q_{i,j} = U_i U_j + U_{i-1} U_{j-1} + \cdots + U_1 U_{j-i+1}.$$
We define now the following matrices ($k \geq 1$ arbitrary integer; $i = 1, 2, \ldots, r$)

\[
C_k = \begin{pmatrix}
U_1 & 0 & 0 & \cdots & 0 \\
U_2 & 1 & 0 & \cdots & 0 \\
U_3 & 0 & 1 & \cdots & 0 \\
& \vdots & \vdots & \ddots & \vdots \\
U_k & 0 & 0 & \cdots & 1
\end{pmatrix} ; \quad A_i = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
a_1 & 1 & 0 & \cdots & 0 \\
a_2 & 0 & 1 & \cdots & 0 \\
& \vdots & \vdots & \ddots & \vdots \\
a_i & 0 & 0 & \cdots & 1
\end{pmatrix} ;
\]

\[
S_{i-r} = I_{r-i} \oplus A_i \text{ (of order } (r+1) \times (r+1));
\]

\[
S_{i-r-1} = S_0 \oplus I_{i-r-1}, l \geq r+1 \text{ (of order } l \times l);
\]

\[
\overline{U}_n = [U_1] \oplus U_{n-1};
\]

\[
G_0 = I_n; G_i = I_{n-i} \oplus A_{i-1}, i = 1, \ldots, r \text{ and }
\]

\[
G_i = I_{n-i} \oplus S_{i-r-1}, l \geq r+1, \text{ where } n \text{ is fixed.}
\]

The inverse of the companion matrix $C_k$ was already computed by Corollary 6.

**Theorem 13.** We have the following factorizations

\[
\overline{U}_k S_{k-r-1} = \overline{U}_k, \quad k \geq r+1,
\]

\[
\overline{U}_i A_{i-1} = \overline{U}_i, \quad i \leq r,
\]

\[
\overline{U}_n = U_1 G_1 G_2 \cdots G_n.
\]

**Proof.** The first factorization follows by using the observation that the multiplication $([U_1] \oplus U_{k-1})(S_0 \oplus U_{k-r-1})$ renders immediately the entries of $\overline{U}_k$ which are not on the first column. The entry on the first column in position $(j, 1)$ is $a_1 U_j + a_2 U_{j-1} + \cdots + a_r U_{j-r}$ which is certainly $U_{j+1}$ by using the recurrence on the sequence $U$. The second factorization follows similarly.

By the first part of the theorem, if $n$ is fixed, then $\overline{U}_n = \overline{U}_n S_{n-r-1} = \overline{U}_n G_n$. Now, $\overline{U}_n = [U_1] \oplus U_{n-1} = [U_1] \oplus \overline{U}_{n-1} S_{n-r-2} = (U_1 I_2 \oplus \overline{U}_{n-1})(I_2 \oplus S_{n-r-2}) = (U_1 I_2 \oplus \overline{U}_{n-1})G_{n-1}$. Continuing in this manner we obtain

\[
\overline{U}_n = (U_1 I_{n-r-1} \oplus \overline{U}_{r+1})G_{r+1} \cdots G_n.
\]

Next, $U_1 I_{n-r-1} \oplus \overline{U}_{r+1} = U_1 I_{n-r} \oplus \overline{U}_r = U_1 I_{n-r} \oplus \overline{U}_r A_{r-1} = (U_1 I_{n-r} \oplus \overline{U}_r)(I_{n-r} \oplus A_{r-1}) = (U_1 I_{n-r} \oplus \overline{U}_r)G_r$. Iterating the procedure and using $\overline{U}_i A_{i-1} = \overline{U}_i$, for $i \leq r$ we get the factorization. \hfill \Box

Finally, Theorem 8 can be further generalized into (the proof is omitted)

**Theorem 14.** The Cholesky factorization of $S_n(U)$ is

\[
S_n(U) = U_n U_n^T .
\]

Our proofs were more straightforward (we believe) because we disregarded some of the properties of the entries of various matrices considered in the previously mentioned research, since these properties are not relevant in this context.
6 Further Remarks

Let \( a = [a_1, a_2, \ldots, a_n] \) and \( \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n] \) be two increasing sequences. The sequence \( a \) is said to be majorized by \( \lambda \), denoted by \( a \prec \lambda \), if

\[
\begin{align*}
\sum_{i=1}^{k} a_i &\leq \sum_{i=1}^{k} \lambda_i, \quad k = 1, \ldots, n - 1 \\
\sum_{i=1}^{n} a_i &= \sum_{i=1}^{n} \lambda_i.
\end{align*}
\]

The well known Schur-Horn Theorem [7] states that if \( H \) is a Hermitian matrix, and \( a \) and \( \lambda \) contain the diagonal entries and eigenvalues of \( H \), respectively, then \( a \prec \lambda \). Using this result, by taking \( a = [U_1^2, U_2^2 + U_2^2, \ldots, \sum_{i=1}^{n} U_i^2] \) and \( \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n] \), the eigenvalues of \( U_n \) in increasing order, we obtain the following result. (Certainly, over \( \mathbb{R} \), a symmetric matrix is Hermitian.)

**Theorem 15.** The eigenvalues of \( U_n \) satisfy

\[
\begin{align*}
\sum_{i=1}^{k} U_i^2 &\leq \sum_{i=1}^{k} \lambda_i, \quad k = 1, \ldots, n - 1; \\
\sum_{i=1}^{k} U_i^2 &= \sum_{i=1}^{k} \lambda_i.
\end{align*}
\]

References


