# Spectral Properties of Some Combinatorial Matrices

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**Abstract**

In this paper we investigate the spectra and related questions for various combinatorial matrices, generalizing work by Carlitz, Cooper and Kennedy.

## 1 Introduction

In [8], R. Peele and P. Stănică studied $n \times n$ matrices with the $(i,j)$ entry the binomial coefficient $\binom{i-1}{j-1}$, respectively, $\binom{i-1}{n-j}$ and derived many interesting results on powers of these matrices. In [10], one of us found that the same is true for a much larger class of what he called *netted matrices*, namely matrices with entries satisfying a certain type of recurrence among the entries of all $2 \times 2$ cells.

Let $R_n$ be the matrix whose $(i,j)$ entries are $a_{i,j} = \binom{i-1}{n-j}$, which satisfy

$$a_{i,j-1} = a_{i-1,j-1} + a_{i-1,j}. \tag{1}$$

The previous recurrence can be extended for $i \geq 0, j \geq 0$, using the boundary conditions $a_{1,n} = 1$, $a_{1,j} = 0$, $j \neq n$. Remark the following consequences of the boundary conditions and recurrence (1): $a_{i,j} = 0$ for $i + j \leq n$, and $a_{i,n+1} = 0, 1 \leq i \leq n$.

The matrix $R_n$ was firstly studied by Carlitz [2] who gave explicit forms for the eigenvalues of $R_n$. Let $f_{n+1}(x) = \det(xI - R_n)$ be the characteristic polynomial of $R_n$. Thus

$$f_n(x) = \sum_{r=0}^{n+1} (-1)^{r(r+1)/2} \binom{n}{r}_F x^{n-r}$$

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where $\binom{n}{r}_F$ denote the Fibonomial coefficient, defined (for $n \geq r > 0$) by

$$\binom{n}{r}_F = \frac{F_1 F_2 \dots F_n}{(F_1 F_2 \dots F_r)(F_1 F_2 \dots F_{n-r})}$$

with $\binom{n}{n}_F = \binom{n}{0}_F = 1$. Carlitz showed that

$$f_n(x) = \prod_{j=0}^{n-1} \left( x - \phi^j \bar{\phi}^{n-j} \right)$$

where $\phi, \bar{\phi} = \left(1 \pm \sqrt{5}\right)/2$. Thus the eigenvalues of $R_n$ are $\phi^n, \phi^{n-1}\bar{\phi}, \dots, \phi\bar{\phi}^{n-1}, \bar{\phi}^n$. We shall give another proof of this result in the next section.

In [8] it was proved that the entries of the power $R_n^e$ satisfy the recurrence

$$F_{e-1} a_{i,j}^{(e)} = F_e a_{i-1,j}^{(e)} + F_{e+1} a_{i-1,j-1}^{(e)} - F_e a_{i,j-1}^{(e)}, \tag{2}$$

where $F_e$ is the Fibonacci sequence. Closed forms for *all* entries of $R_n^e$ were not found, but several results concerning the generating functions of rows and columns were obtained (see [8, 10]). For instance, the entries in the first row and column of $R_n^e$ are

$$\begin{aligned} a_{1,j}^{(e)} &= \binom{n-1}{j-1} F_{e-1}^{n-j} F_e^{j-1} \\ a_{i,1}^{(e)} &= F_{e-1}^{n-i} F_e^{i-1}. \end{aligned} \tag{3}$$

Further, the generating function for the $(i, j)$-th entry of the $e$-th power of a generalization of $R_n$, namely

$$Q_n(a,b) = \left( a^{i+j-n-1}\, b^{n-j} \binom{i-1}{n-j} \right)_{1 \leq i,j \leq n}$$

is

$$B_n^{(e)}(x,y) = \frac{(U_{e-1} + U_e y)(b\, U_{e-1} + y\, U_e)^{n-1}}{U_{e-1} + U_e y - x(U_e + U_{e+1} y)}.$$

Certainly, $Q_n(1,1) = R_n$. As an example,

$$Q_6(a,b) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & b & a \\ 0 & 0 & 0 & b^2 & 2ab & a^2 \\ 0 & 0 & b^3 & 3ab^2 & 3a^2b & a^3 \\ 0 & b^4 & 4ab^3 & 6a^2b^2 & 4a^3b & a^4 \\ b^5 & 5ab^4 & 10a^2b^3 & 10a^3b^2 & 5a^4b & a^5 \end{pmatrix}.$$

Regarding this generalization, in [3], the authors gave the characteristic polynomial of $Q_n(a,b)$ and the trace of $k$th power of $Q_n(a,b,)$, that is, $\operatorname{tr}\left(Q_n^k(a,b)\right)$, by using the method of Carlitz [2].

**Lemma 1.** *Let* $g_n(x)$ *denote the characteristic polynomial of* $Q_n(a,b)$. *Thus*

$$g_n(x) = \sum_{i=0}^{n} (-1)^{i(i+1)/2} b^{i(i-1)/2} \binom{n}{i}_U x^{n-i} \tag{4}$$

*and*

$$tr\left(Q_n^k(a,b)\right) = \frac{U_{kn}}{U_k}, \tag{5}$$

*where* $\binom{n}{i}_U$ *stands for the generalized Fibonomial coefficient, defined by*

$$\binom{n}{r}_U = \frac{U_1U_2\ldots U_n}{(U_1U_2\ldots U_r)(U_1U_2\ldots U_{n-r})},$$

*for* $n \geq i > 0$, *where* $\binom{n}{n}_U = \binom{n}{0}_U = 1$.

One of us extended in [10] some of these results to netted matrices, say $R_n^{\alpha,\beta,\gamma,\delta}$, whose entries $a_{i,j}$, $i \geq 0$, $j \geq 0$ satisfy (for $i \geq 1, j \geq 1$)

$$\delta a_{i,j} = \alpha a_{i-1,j} + \beta a_{i-1,j-1} + \gamma a_{i,j-1}, \tag{6}$$

with the boundary conditions

$$\beta a_{i,0} + \gamma a_{i+1,0} = 0, \text{ for all } 1 \leq i \leq n-1 \tag{7}$$

$$\delta a_{i+1,n+1} - \alpha a_{i,n+1} = 0, \text{ for all } 1 \leq i \leq n-1. \tag{8}$$

It was shown in [10] that the entries of any power of such a matrix will satisfy a similar recurrence, and generating functions for these powers were found. In the case of the Fibonacci sequence, the generating function of the $i$-th row of $R_n^e$ is

$$r_i^{(e)}(x) = \sum_{j\geq 1} a_{i,j}^{(e)} x^{j-1} = (F_e + F_{e+1}x)^{i-1}(F_{e-1} + F_e x)^{n-i}$$

and, if $e > 1$, the generating function for the $j$-th column of $R_n^e$ is

$$c_j^{(e)}(x) = \left(\frac{F_{e+1}x - F_e}{F_{e-1} - F_e x}\right)^{j-1} \frac{F_{e-1}^n}{F_{e-1} - F_e x}\left[1 + \sum_{s=1}^{j-1}\binom{n}{s}\left(\frac{F_e(F_{e-1} - F_e x)}{F_{e-1}(F_{e+1}x - F_e)}\right)^s\right].$$

where $c_j^{(1)}(x) = (1-x)^{j-1-n}x^{n-j}$.

The matrix $Q_n(a,b)$ was introduced as a generalization of the Fibonacci matrix $Q = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ which has the property that $Q^n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n-1} \\ F_n \end{pmatrix}$. The matrix $Q_n(a,b)$ displays a similar property on the sequence $U_{n+1} = aU_n + bU_{n-1}$, that is, any $e$-th power of $Q_n(a,b)$ multiplied by a fixed vector gives an $n$-tuple of consecutive terms of the sequence $U = (U_k)_k$. We refer to [10, 3] for more details.

## 2 Spectra of $R_n$ and $Q_n(a,b)$

In [8], the authors proposed a conjecture on the eigenvalues, which was proven independently in [1] and the unpublished manuscript [11]. In this section we give the proof of the conjecture from [11] and we find the eigenvectors of $R_n$. In [8, 10], it was shown that the inverse of $R_n$ is the matrix

$$R_n^{-1} = \left( (-1)^{n+i+j+1} \binom{n-i}{j-1} \right)_{1\le i,j\le n},$$

and, in general, the inverse of $Q_n(a,b)$ is

$$Q_n^{-1}(a,b) = \left( (-1)^{n+i+j+1} a^{n+1-i-j}\, b^{i-n} \binom{n-i}{j-1} \right)_{1\le i,j\le n}. \tag{9}$$

We define $K_n$ to be the matrix with $(i,j)$-entry $\delta_{i,n-j+1}$ (the Kronecker symbol), that is, $K_n$ is a permutation matrix having 1 on the secondary diagonal and 0 elsewhere.

**Theorem 2.** *Let $\phi = \frac{1+\sqrt{5}}{2}, \bar{\phi} = \frac{1-\sqrt{5}}{2}$ be the golden section and its conjugate. The eigenvalues of $R_n$ are:*
*1. $\{(-1)^{k+i}\phi^{2i-1}, (-1)^{k+i}\bar{\phi}^{2i-1}\}_{i=1,\ldots,k}$, if $n=2k$.*
*2. $\{(-1)^k\} \cup \{(-1)^{k+i}\phi^{2i}, (-1)^{k+i}\bar{\phi}^{2i}\}_{i=1,\ldots,k}$, if $n=2k+1$.*
*Equivalently, the eigenvalues of $R_n$ are $\phi^n, \phi^{n-1}\bar{\phi}, \ldots, \phi\bar{\phi}^{n-1}, \bar{\phi}^n$.*

*Proof.* First, we show that $R_n$ is a permutation matrix away from $L_n$, namely

$$R_n \cdot K_n = L_n \iff R_n = L_n \cdot K_n. \tag{10}$$

It is a trivial matter to prove $K_n^2 = I_n$, which will give the equivalence. It suffices to show the second identity, which follows easily, since an entry in $L_n \cdot K_n$ is

$$\sum_{k=1}^{n} \binom{i-1}{k-1} \delta_{k,n-j+1} = \binom{i-1}{n-j}.$$

Now, denote by $A_n$, the matrix obtained by taking absolute values of entries of $R_n^{-1}$. We show that $R_n$ has the same characteristic polynomial (eigenvalues) as $A_n$, namely we prove their similarity,

$$K_n \cdot R_n \cdot K_n = A_n. \tag{11}$$

Since $K_n \cdot R_n \cdot K_n = K_n \cdot L_n$ (by (10)), to show (11) it suffices to prove that $K_n \cdot L_n = \left(\binom{n-i}{j-1}\right)_{i,j}$. Therefore, we need

$$\sum_{k=1}^{n} \delta_{i,n-k+1} \binom{k-1}{j-1} = \binom{n-i}{j-1},$$

which is certainly true.

We use a result of [7] to show that $A_n$ in turn is similar to $D_n$, the diagonal matrix whose diagonal entries are the elements in the eigenvalues set listed in decreasing order according to size of the absolute value. For instance, for $n = 4$,

$$D_4 = \begin{pmatrix} \alpha^3 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & \beta^3 \end{pmatrix}$$

Our claim will be proved if one can show that $A_n$ is similar to $D_n$, and this will follow from [7]. We sketch here that argument for the convenience of the reader. Define an array $b_{n,m}$ by

$$\begin{aligned} b_{n,0} &= 1 \text{ for all } n \geq 0 \\ b_{n,m} &= 0 \text{ for all } m > n \\ b_{n,m} &= b_{n-1,m-1}\,(-1)^m \frac{F_n}{F_m} \text{ for all } m \leq n \end{aligned}$$

Certainly, $|b_{n,m}| = \dfrac{F_n F_{n-1} \cdots F_{n-m+1}}{F_m \cdots F_1}$. Let $C_n = (c_{i,j})_{i,j}$, where

$$\begin{cases} c_{i,i+1} = 1 & if\, i = 1, \ldots, n-1 \\ c_{n,j} = -b_{n,n+1-j} & if\, j = 1, \ldots, n \\ c_{i,j} = 0 & otherwise. \end{cases}$$

We observe that, in fact, $C_n$ is the companion matrix of the polynomial with coefficients $b_{n,n+1-j}$. Let $X_n$ be the matrix with entries $\binom{n-i}{j-1} F_{i-2}^{j-1} F_{i-1}^{n-j}$. It turns out that the eigenvector matrix $E_n$ of $A_n$, with columns vectors listed in decreasing order of absolute value of the corresponding eigenvalues, normalized so that the last row is made up of all 1's, satisfies

$$X_n E_n = V_n,$$

where $V_n$ is the Vandermonde matrix, which is the eigenvector matrix of $C_n$ with eigenvectors listed in decreasing order of the absolute values of the corresponding eigenvalues. Also,

$$X_n A_n X_n^{-1} = C_n \text{ and } E_n^{-1} A_n E_n = D_n.$$

Our theorem follows. ∎

Easily we deduce

**Corollary 3.** *The eigenvectors matrix of $R_n$, say $W_n$, with eigenvectors listed in decreasing order of the absolute values of the corresponding eigenvalues, is*

$$W_n = K_n E_n = K_n X_n^{-1} V_n.$$

*Proof.* We showed that $K_n R_n K_n = A_n$ and $E_n^{-1} A_n E_n = D_n$. It follows that $(E_n^{-1} K_n) R_n (K_n E_n) = D_n$, which together with $X_n E_n = V_n$, proves the corollary. ∎

**Example 4.** *For $n = 4$, the eigenvectors matrix is*

$$W_4 = \begin{pmatrix} -\alpha^3 & \alpha & \beta & -\beta^3 \\ \alpha^2 & -\frac{1}{3}\beta & -\frac{1}{3}\alpha & \beta^2 \\ -\alpha & -\frac{1}{3}\alpha^2 & -\frac{1}{3}\beta^2 & -\beta \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

We present below the first few characteristic polynomials of $Q_n(a,b)$ (for $1 \le n \le 7$) and we ask whether one can find their roots as in Theorem 2:

$$\begin{aligned} & x^2 - ax - b; \\ & -(b+x)\left(-xa^2 + b^2 + x^2 - 2bx\right); \\ & \left(b^3 - axb - x^2\right)\left(xa^3 + 3bxa + b^3 - x^2\right); \\ & \left(b^2 - x\right)\left(-xa^4 - 4bxa^2 + b^4 + x^2 - 2b^2x\right)\left(b^4 + 2xb^2 + a^2xb + x^2\right); \\ & -\left(b^5 - 3axb^2 - a^3xb - x^2\right)\left(b^5 + axb^2 - x^2\right)\left(xa^5 + 5bxa^3 + 5b^2xa + b^5 - x^2\right). \end{aligned}$$

The argument that proves the next theorem is similar to the proof of Theorem 2, and so, we omit it. One can also find two different ways to see its proof in [3] and [4].

**Theorem 5.** *Let $\lambda = \frac{a+\sqrt{a^2+4b}}{2}, \bar{\lambda} = \frac{a-\sqrt{a^2+4b}}{2}$ be the roots of the polynomial $x^2 - ax - b$. The eigenvalues of $Q_n(a,b)$ are:*
*1. $\{(-b)^{k-i}\lambda^{2i-1}, (-b)^{k-i}\bar{\lambda}^{2i-1}\}_{i=1,\dots,k}$, if $n = 2k$.*
*2. $\{(-b)^k\} \cup \{(-b)^{k-i}\lambda^{2i}, (-b)^{k-i}\bar{\lambda}^{2i}\}_{i=1,\dots,k}$, if $n = 2k+1$. Equivalently, the eigenvalues of $Q_n(a,b)$ can also be written in the following compact form $\lambda^n,\ \lambda^{n-1}\bar{\lambda}, \dots \lambda\bar{\lambda}^{n-1}, \bar{\lambda}^n$.*

Since $R_n, Q_n(a,b)$ are column justified triangular matrices, the determinants of our matrices are easy to compute, or we can use the previous result.

**Corollary 6.** *We have* $\det(R_n) = (-1)^{\lfloor n/2 \rfloor}$ *and* $\det(Q_n(a,b)) = (-1)^{\lfloor n/2 \rfloor} b^{n(n-1)/2}$.

# 3 Arithmetic progressions and spectra of other combinatorial matrices

Let the sequences $\{u_n\}$, $\{v_n\}$ be defined by

$$\begin{aligned} u_n &= au_{n-1} + bu_{n-2} \\ v_n &= av_{n-1} + bv_{n-2}, \end{aligned}$$

for $n > 1$, where $u_0 = 0, u_1 = 1$, and $v_0 = 2, v_1 = a$, respectively. Let $\alpha, \beta$ be the roots of the associated equation $x^2 - ax - b = 0$. The next lemma appears in [6].

**Lemma 7.** *For $k \ge 1$ and $n > 1$,*

$$\begin{aligned} u_{kn} &= v_k u_{k(n-1)} + (-1)^{k+1} b^k u_{k(n-2)} \\ v_{kn} &= v_k v_{k(n-1)} + (-1)^{k+1} b^k v_{k(n-2)}. \end{aligned} \tag{12}$$

Using the sequence $v_k$, we define the $n\times n$ matrix $H_n\left(v_k, b^k\right)$ as follows:

$$H_n\left(v_k, b^k\right) = \left(v_k^{i+j-n-1}\left(-\left(-b\right)^k\right)^{n-j}\binom{i-1}{n-j}\right)_{1\le i,j\le n}.$$

Observe that $H_n\left(v_1, b^1\right) = Q_n\left(a, b\right)$. As an example,

$$H_6(v_k, b^k) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\left(-b\right)^k & v_k \\ 0 & 0 & 0 & b^{2k} & -2\left(-b\right)^k v_k & v_k^2 \\ 0 & 0 & -\left(-b\right)^{3k} & 3b^{2k}v_k & -3\left(-b\right)^k v_k^2 & v_k^3 \\ 0 & b^{4k} & -4\left(-b\right)^{3k} v_k & 6b^{2k}v_k^2 & -4\left(-b\right)^k v_k^3 & v_k^4 \\ -\left(-b\right)^{5k} & 5b^{4k}v_k & -10\left(-b\right)^{3k} v_k^2 & 10b^{2k}v_k^3 & -5\left(-b\right)^k v_k^4 & v_k^5 \end{pmatrix}.$$

As in equation (9), we can easily find the inverse of the matrix $H_n$, namely

$$H_n^{-1}(v_k, b^k) = \left((-1)^{j+1}(-b)^{-k(n-i)}v^{n+1-i-j}\begin{pmatrix} n-i \\ j-1 \end{pmatrix}\right)_{i,j}.$$

It is well known that for $n \ge -1$,

$$u_{n+1} = \sum_r \binom{r}{n-r} a^{2r-n}b^{n-r}. \tag{13}$$

We generalize this identity next.

**Lemma 8.** *For $k > 0$ and $n \ge -1$,*

$$\frac{u_{k(n+1)}}{u_k} = \sum_r \binom{r}{n-r} v_k^{2r-n}\left(-\left(-b\right)^k\right)^{n-r}.$$

*Proof.* (Induction on $n$) If $n = 1$, then equations (12) and (13) give the identity. Suppose that the result is true for $n-1$ and $n$. Then

$$\begin{aligned} u_{k(n+1)} &= v_k u_{kn} + \left(-b\right)^{k+1} u_{k(n-1)} \\ &= v_k \sum_r \binom{r}{n-1-r} v_k^{2r-n+1}\left(-\left(-b\right)^k\right)^{n-1-r} \\ &\quad -\left(-b\right)^k \sum_r \binom{r}{n-2-r} v_k^{2r-n+2}\left(-\left(-b\right)^k\right)^{n-2-r} \\ &= \sum_r \left[\binom{r}{n-1-r} + \binom{r}{n-2-r}\right] v_k^{2r-n+2}\left(-\left(-b\right)^k\right)^{n-1-r} \\ &= \sum_r \binom{r+1}{n-1-r} v_k^{2r-n+2}\left(-\left(-b\right)^k\right)^{n-1-r} \\ &= \sum_r \binom{r}{n-r} v_k^{2r-n}\left(-\left(-b\right)^k\right)^{n-r}. \end{aligned}$$

∎

**Lemma 9.** *Let $k, m > 0$ and $0 \leq r \leq n$. Then*

$$\begin{aligned}
&\left(\frac{u_{km}x - (-b)^m u_{(k-1)m}}{u_m}\right)^r \left(\frac{u_{(k+1)m}x - (-b)^m u_{km}}{u_m}\right)^{n-r} \\
= &\sum_{r_1, r_2, \ldots, r_k} \binom{n-r}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_{k-1}}{r_k} \\
&\times v_m^{kn-r-2r_1-\cdots-2r_{k-1}-r_k} \left(-(-b)^m\right)^{r_1+r_2+\ldots+r_k} x^{n-r_k}. \qquad (14)
\end{aligned}$$

*Proof.* (Induction on $k$) For $k = 1$, the proof follows from $u_{2n} = u_n v_n$ and

$$x^r \left(v_m x - (-b)^m\right)^{n-r} = \sum_s \binom{n-r}{s} v_m^{n-r-s} \left(-(-b)^m\right)^s x^{n-s}$$

(see [3]). We assume that the equality holds for some positive integer $k$. If we replace $\left(v_m - (-b)^m x^{-1}\right)$ by $x$ and then multiply by $x^n$, the left side of this equation becomes

$$\left(\frac{u_{(k+1)m}x - (-b)^m u_{km}}{u_m}\right)^r \left(\frac{u_{(k+2)m}x - (-b)^m u_{(k+1)m}}{u_m}\right)^{n-r}.$$

After some simplifications, if we expand the right side of this equation, then we get

$$\begin{aligned}
&\sum_{r_1, r_2, \ldots, r_{k+1}} \binom{n-r}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_k}{r_{k+1}} \\
&\times v_m^{k(n+1)-r-2r_1-\cdots-2r_k-r_{k+1}} \left(-(-b)^m\right)^{r_1+r_2+\cdots+r_{k+1}} x^{n-r_{k+1}}.
\end{aligned}$$

The proof is complete. ∎

**Lemma 10.** *For all $m > 0$,*

$$tr\left(H_n^m\left(v_k, b^k\right)\right) = \frac{u_{knm}}{u_k}.$$

*Proof.* If we multiply both sides of the equation (14) by $x^r$ and summing over $r$, we get

$$\begin{aligned}
&\sum_{r=0}^{n} \left(\frac{u_{km}x - (-b)^m u_{(k-1)m}}{u_m}\right)^r \left(\frac{u_{(k+1)m}x - (-b)^m u_{km}}{u_m}\right)^{n-r} x^r \\
= &\sum_{r_1, r_2, \ldots, r_k} \binom{n-r}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_{k-1}}{r_k} \\
&\times v_m^{kn-r-2r_1-\cdots-2r_{k-1}-r_k} \left(-(-b)^m\right)^{r_1+r_2+\cdots+r_k} x^{n+r-r_k},
\end{aligned}$$

where the coefficient of $x^n$ on the right side is $\mathrm{tr}\left(H_n^m\left(v_k, b^k\right)\right)$. The coefficient of $x^n$ on the left side of this equation without its denominator is

$$\begin{aligned}
&\sum_{r+s+t=n} \binom{r}{s}\binom{n-r}{t} u_{km}^s \left(-(-b)^m u_{(k-1)m}\right)^{r-s} u_{(k+1)m}^t \left(-(-b)^m u_{km}\right)^s \\
= &\sum_{r+s\leq n} \binom{r}{s}\binom{n-r}{s} u_{km}^s \left(-(-b)^m u_{(k-1)m}\right)^{r-s} u_{(k+1)m}^{n-r-s} \left(-(-b)^m u_{km}\right)^s.
\end{aligned}$$

For easy writing, denote this last expression by $c_n$. Then

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_n x^n \\
=\;& \sum_{r,s=0}^{\infty} \binom{r}{s} \left(-(-b)^m\right)^r u_{(k-1)m}^{r-s} u_{km}^{2s} x^{r+s} \sum_{n=r+s} \binom{n-r}{s} \left(u_{(k+1)m}x\right)^{n-r-s} \\
=\;& \sum_{r,s=0}^{\infty} \binom{r}{s} \left(-(-b)^m\right)^r u_{(k-1)m}^{r-s} u_{km}^{2s} x^{r+s} \left(1-u_{(k+1)m}x\right)^{-s-1} \\
=\;& \sum_{s=0}^{\infty} \left(-(-b)^m\right)^s u_{km}^{2s} x^{2s} \left(1-u_{(k+1)m}x\right)^{-s-1} \sum_{r\geq s} \binom{r}{s} \left(\left(-(-b)^m\right) u_{(k-1)m}x\right)^{r-s} \\
=\;& \sum_{s=0}^{\infty} \left(-(-b)^m\right)^s u_{km}^{2s} x^{2s} \left(1-u_{(k+1)m}x\right)^{-s-1} \left(1+(-b)^m u_{(k-1)m}x\right)^{-s-1} \\
=\;& \frac{1}{\left(1-u_{(k+1)m}x\right)\left(1+(-b)^m u_{(k-1)m}x\right)} \frac{1}{1-\frac{\left(-(-b)^m\right)^s u_{km}^{2s} x^2}{\left(1-u_{(k+1)m}x\right)\left(1+(-b)^m u_{(k-1)m}x\right)}} \\
=\;& \frac{1}{\left(1-u_{(k+1)m}x\right)\left(1+(-b)^m u_{(k-1)m}x\right)+(-b)^m u_{km}^{2s} x^2}. \qquad (15)
\end{aligned}$$

Here by the Binet formula for $\{u_n\}$, we note that for all integers $k>0$,

$$\frac{u_{kn}-(-b)^k u_{k(n-2)}}{u_k} = v_{k(n-1)} \qquad (16)$$

and

$$-\left(\frac{u_{kn}u_{k(n-2)}-u_{k(n-1)}^2}{u_k^2}\right) = (-b)^{k(n-2)}. \qquad (17)$$

Using (16) and (17), we write the right side of the equation (15) as

$$\begin{aligned}
&\frac{1}{\left(1-u_{(k+1)m}x\right)\left(1+(-b)^m u_{(k-1)m}x\right)+(-b)^m u_{km}^{2s} x^2} \\
=\;& \frac{1}{1-v_{mk}x+(-b)^{mk}x^2}.
\end{aligned}$$

Thus, $c_n = \frac{u_{kmn}}{u_k}$, and so, $\operatorname{tr}\left(H_n^m\left(v_k, b^k\right)\right) = \frac{u_{knm}}{u_k}$. The proof is complete. ∎

**Theorem 11.** *The eigenvalues of* $H_n\left(v_k, b^k\right)$ *are*

$$\alpha^{kn}, \alpha^{k(n-1)}\beta^k, \ldots, \alpha^k\beta^{k(n-1)}, \beta^{kn}.$$

*Proof.* Let $f_n(x) = \det\left(xI - H_n\left(v_k, b^k\right)\right)$ and $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ denote the eigenvalues of $H_n\left(v_k, b^k\right)$. Then by Lemma 10,

$$\begin{aligned}
\frac{f_n'(x)}{f_n(x)} &= \sum_{j=0}^{n-1} \frac{1}{x-\lambda_j} = \sum_{m=0}^{n-1} x^{-m-1} \sum_{j=0}^{m} \lambda_j^m \\
&= \sum_{k=0}^{\infty} x^{-k-1} \operatorname{tr}\left(H_{n-1}^m\left(v_k, b^k\right)\right) \\
&= \sum_{m=0}^{\infty} x^{-m-1} \frac{u_{k(n-1)m}}{u_k} \\
&= \sum_{m=0}^{\infty} x^{-m-1} \sum_{j=0}^{n-1} \alpha^{jkm} \beta^{k(n-j-1)m} \\
&= \sum_{j=0}^{n-1} \frac{1}{x - \alpha^{jk}\beta^{(n-j-1)k}}.
\end{aligned}$$

Thus

$$f_n(x) = \prod_{j=0}^{n-1}\left(x - \alpha^{jk}\beta^{(n-j-1)k}\right)$$

and so the eigenvalues of $H_n\left(v_k, b^k\right)$ are

$$\alpha^{k(n-1)}, \alpha^{k(n-2)}\beta^k, \ldots, \alpha^k\beta^{k(n-2)}, \beta^{k(n-1)}.$$

∎

Next, we give an expression of the characteristic polynomial of $H_n$ in terms of Gaussian binomials. Fix $k$ and let $r_n := u_{kn}$. Define the $k$-Fibonomial coefficients by

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{u,k} = \frac{r_1 r_2 \ldots r_n}{\left(r_1 r_2 \ldots r_m\right)\left(r_1 r_2 \ldots r_{n-m}\right)},$$

where $\left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\}_{u,k} = \left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\}_{u,k} = 1$.

Clearly, when $k = 1$, then $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}_{u,k}$ is reduced to the Fibonomial coefficient $\binom{n}{m}_u$.

**Theorem 12.**

$$f_n(x) = \prod_{j=0}^{n-1}\left(x - \alpha^{jk}\beta^{(n-j-1)k}\right) = \sum_{i=0}^{n} (-1)^i (-b)^{ki(i-1)/2} \left\{ \begin{matrix} n \\ i \end{matrix} \right\}_{u,k} x^{n-i}.$$

*Proof.* Here we can use the following familiar identity

$$\prod_{j=0}^{n-1}\left(1 - q^j x\right) = \sum_{i=0}^{n} (-1)^i q^{i(i-1)/2} \begin{bmatrix} n \\ i \end{bmatrix}_q x^i, \tag{18}$$

where

$$\begin{bmatrix} n \\ i \end{bmatrix}_q = \frac{(1-q^n)\dots\left(1-q^{n-i+1}\right)}{(1-q^i)\dots(1-q)},$$

are the usual $q$-binomial coefficients (Gaussian binomials).

If we replace $q$ by $(\beta/\alpha)^k$ we find that for all positive fixed integers $k$,

$$\begin{bmatrix} n \\ i \end{bmatrix}_q \to \alpha^{ki(i-n)} \begin{Bmatrix} n \\ i \end{Bmatrix}_{u,k}.$$

Thus (18) becomes

$$\prod_{j=0}^{n-1}\left(1-\left(\alpha^{-1}\beta\right)^{kj} x\right) = \sum_{i=0}^{n} (-1)^i \beta^{ki(i-1)/2} \alpha^{ki(i+1)/2-nki} \begin{Bmatrix} n \\ i \end{Bmatrix}_{u,k} x^i.$$

Replacing $x$ by $\alpha^{k(n-1)}x$, we get

$$\begin{aligned}\prod_{j=0}^{n-1}\left(1-\alpha^{(n-j-1)k}\beta^{kj}x\right) &= \sum_{i=0}^{n} (-1)^i (\alpha\beta)^{ki(i-1)/2} \begin{Bmatrix} n \\ i \end{Bmatrix}_{u,k} x^i \\ &= \sum_{i=0}^{n} (-1)^i (-b)^{ki(i-1)/2} \begin{Bmatrix} n \\ i \end{Bmatrix}_{u,k} x^i.\end{aligned}$$

Finally replacing $x$ by $x^{-1}$ gives us

$$\prod_{j=0}^{n-1}\left(1-\alpha^{(n-j-1)k}\beta^{kj}x\right) = \sum_{i=0}^{n} (-1)^i (-b)^{ki(i-1)/2} \begin{Bmatrix} n \\ i \end{Bmatrix}_{u,k} x^{n-i}.$$

∎

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