ON THE FIRST DIGITS OF THE FIBONACCI NUMBERS AND THEIR EULER FUNCTION

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ABSTRACT. Here, we show that given any two finite strings of base $b$ digits, say $s_1$ and $s_2$, there are infinitely many Fibonacci numbers $F_n$ such that the base $b$ representation of $F_n$ starts with $s_1$ and the base $b$ representation of $\phi(F_n)$ starts with $s_2$.

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1. Introduction

Let $b \geq 2$ be an integer. Let $s_1 = c_1 \cdots c_k(b)$ be a positive integer $s_1$ written in base $b$. Washington [4] proved that there exist infinitely many Fibonacci numbers $F_n$ whose base $b$ representation starts with $s_1$. In fact, the first digits of the Fibonacci sequence obey Benford’s law in that the proportion of the positive integers $n$ such that $F_n$ starts with $s_1$ is precisely $\log((s_1 + 1)/s_1)/\log b$. Here, we take this one step further. Let $\phi(m)$ be the Euler function of the positive integer $m$. We put $s_2 = d_1 \cdots d_\ell(b)$ for some other positive integer written in base $b$ and prove the following theorem.

**Theorem.** Given positive integers $s_1 = c_1 \cdots c_k(b)$ and $s_2 = d_1 \cdots d_\ell(b)$ written in base $b$, there exist infinitely many positive integers $n$ such that the base $b$ representation of $F_n$ starts with the digits of $s_1$ and the base $b$ representation of $\phi(F_n)$ starts with the digits of $s_2$.

We use the fact that with $\alpha, \beta = ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$ the Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

holds for all $n \geq 0$.

For a positive real number $x$ we write $\log x$ for the natural logarithm of $x$, and $[x]$, respectively, $\{x\}$, for the integer part, respectively, fractional part of $x$.

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2. The proof

By replacing \( s_1 \) with \( s_1 b^m \) for some positive integer \( m \), if needed, whose effect is adding \( m \) zeros at the end of the base \( b \) representation of \( s_1 \), we may assume that \( s_1 > s_2 \). By replacing \( s_1, s_2 \) by \( s_1 b^m \), respectively, \( s_2 b^m \) for an arbitrary positive integer \( m \), we may assume that the length of the base \( b \) representation of \( s_1 \), that is \( k \), is as large as we wish. In Section 4 of [2], it is shown that \( \phi(F_n)/F_n \) is dense in \([0,1]\). So, we take \( \varepsilon \in (0,1/(15b^{2k})) \) and choose a positive integer \( a \) such that

\[
\frac{\phi(F_a)}{F_a} \in \left( \frac{s_2}{s_1} + \varepsilon, \frac{s_2}{s_1} + 2\varepsilon \right). \tag{1}
\]

Now we take any prime \( p > F_a \) and look at \( F_{ap} \). Since \( p > F_a \), it follows that

\[ F_{ap} = F_a \left( \frac{F_{ap}}{F_a} \right), \]

and the two factors \( F_a \) and \( F_{ap}/F_a \) on the right above are coprime (indeed, the only common prime factor of these two numbers could be \( p \), which is not the case since \( p > F_a \)). Any prime factor \( q \) of \( F_{ap}/F_a \) is a primitive prime factor of \( F_{dp} \) for some divisor \( d \) of \( a \). Recall that a prime number \( q \) is said to be a primitive prime factor of \( F_n \) if \( q \) divides \( F_n \), but does not divide any \( F_m \) for \( 1 \leq m < n \). One of the properties of primitive prime factors \( q \) of \( F_n \) when \( n > 5 \) is that \( q \equiv \pm 1 \pmod{n} \). In particular, every prime factor \( q \) of \( F_{ap}/F_a \) is congruent to \( \pm 1 \pmod{p} \).

Let \( q_1, \ldots, q_t \) be all the prime factors of \( F_{ap}/F_a \). Then

\[(2p - 1)^t \leq q_1 \cdots q_t \leq \frac{F_{ap}}{F_a} \leq F_{ap} \leq \alpha^{ap}.
\]

Thus, \( t = O(p/\log p) \). Then

\[
\frac{\phi(F_{ap})}{F_{ap}} = \left( \frac{\phi(F_a)}{F_a} \right) \prod_{i=1}^t \left( 1 - \frac{1}{q_i} \right)
= \frac{\phi(F_a)}{F_a} \exp \left( - \sum_{i=1}^t \frac{1}{q_i} + O \left( \sum_{q \geq q_1} \frac{1}{q^2} \right) \right)
= \frac{\phi(F_a)}{F_a} \exp \left( O \left( \frac{t}{q_1} \right) \right)
= \frac{\phi(F_a)}{F_a} \exp \left( O \left( \frac{1}{\log p} \right) \right)
\]
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\[
= \frac{\phi(F_a)}{F_a} \left( 1 + O \left( \frac{1}{\log p} \right) \right).
\]

It implies that, if \( p > \exp(\kappa \varepsilon^{-1}) \), where \( \kappa > 0 \) is some absolute constant, then

\[
\frac{\phi(F_{ap})}{F_{ap}} \in \left( \frac{s_2}{s_1} + 0.5\varepsilon, \frac{s_2}{s_1} + 1.5\varepsilon \right).
\]

(2)

We now follow Washington’s argument [4] to prove that there exist infinitely many primes \( p \) such that the base \( b \) representation of \( F_{ap} \) starts with \( s_1 \). For this, it is enough to show that

\[
F_{ap} = s_1 b^N + \zeta_{ap} \quad \text{for some integer} \quad 0 \leq \zeta_{ap} \leq b^N - 1.
\]

(3)

Note that since \( q_1 \geq 2p - 1 \), it follows that if \( p \) is sufficiently large (say, \( p > b^k \)), then \( F_{ap} \) cannot equal \( s_1 b^N \), and in particular, if in the above formula (3) we have \( \zeta_{ap} \geq 0 \), then in fact \( \zeta_{ap} \geq 1 \). The above formula (3) yields

\[
\alpha_{ap} = \sqrt{5}s_1 b^N + \sqrt{5}\zeta_{ap} + \beta_{ap} = \sqrt{5}s_1 b^N (1 + x_{ap}).
\]

Since \( \zeta_{ap} \geq 1 \), it follows that \( \sqrt{5}\zeta_{ap} + \beta_{ap} > \sqrt{5} - 1 > 1 \), and

\[
0 < x_{ap} = \frac{\sqrt{5}\zeta_{ap} + \beta_{ap}}{\sqrt{5}s_1 b^N}.
\]

So, if \( x_{ap} \in (0,1/b^k) \), and \( p > b^k \) is sufficiently large, it then follows that \( \zeta_{ap} < b^N \), which is what we want. Thus,

\[
ap \log \alpha = \log(\sqrt{5}s_1) + N \log b + \log(1 + x_{ap}),
\]

or

\[
ap \log \alpha = \frac{\log(\sqrt{5}s_1)}{\log b} - \left\lfloor \frac{\log(\sqrt{5}s_1)}{\log b} \right\rfloor + \log(1 + x_{ap}) + \frac{\log(1 + x_{ap})}{\log b}.
\]

(4)

Observe that \( \log(\sqrt{5}s_1)/\log b \) is never an integer. Assume that \( k \) is sufficiently large such that

\[
\frac{1}{b^k \log b} < 1 - \left\lfloor \frac{\log(\sqrt{5}s_1)}{\log b} \right\rfloor.
\]

Then putting

\[
\delta = \frac{\log(1 + 1/b^k)}{\log b},
\]

we see that a relation like (4) with \( x_{ap} \in (0,1/b^k) \) holds provided that

\[
\left\{ p \left( \frac{a \log \alpha}{\log b} \right) \right\} \in \left( \left\{ \frac{\log(\sqrt{5}s_1)}{\log b} \right\}, \left\{ \frac{\log(\sqrt{5}s_1)}{\log b} \right\} + \delta \right).
\]

(5)
The number \( \gamma = a \log \alpha / \log b \) is irrational. By a result of Vinogradov \[3\], the sequence of fractional parts \( \{p\gamma\}_{p \text{ prime}} \) is uniformly distributed. In particular, containment \[5\] holds for a positive proportion of primes \( p \), and therefore certainly for infinitely many of them. So, indeed relation \[3\] holds. Relation \[2\] now shows that

\[
\phi(F_{ap}) = s_2 b^N + \theta,
\]

where

\[
\theta \in \left( \zeta_{ap} \left( \frac{s_2}{s_1} + 0.5\varepsilon \right) + 0.5\varepsilon b^N, \zeta_{ap} \left( \frac{s_2}{s_1} + 1.5\varepsilon \right) + 1.5\varepsilon s_1 b^N \right).
\]

Since \( \varepsilon < 1/(15b^{2k}) \), the above upper bound is

\[
\zeta_{ap} \left( \frac{s_2}{s_1} + 1.5\varepsilon \right) + 1.5\varepsilon s_1 b^N \leq (b^N - 1) \left( \frac{b^k - 1}{b^k} + \frac{0.1}{b^k} \right) + \frac{0.1(b^k - 1)}{b^{2k}} b^N
\]

\[
< b^N - 1,
\]

where the last inequality above is implied by

\[
\frac{1}{9} < \frac{b^N - 1}{b^N},
\]

which holds true for all \( b \geq 2 \) and \( N \geq 1 \). This completes the proof of the theorem.

### 3. Comments

It was shown in \[1\] that with \( \sigma(m) \) being the sum of divisors of the positive integer \( m \), the ratio \( \sigma(F_n)/F_n \) is dense in \([1, \infty)\). The present method now shows that there are infinitely many positive integers \( n \) such that the base \( b \) representation of \( F_n \) starts with the digits of \( s_1 \) and the base \( b \) representation of \( \sigma(F_n) \) starts with the digits of \( s_2 \). Also, one may replace the Fibonacci sequence \( F_n \) in the above statements with some other sequence \( u_n \) for which it has been proved that \( \phi(u_n)/u_n \) and \( u_n/\sigma(u_n) \), respectively, are dense in \([0, 1]\). For example, one can take \( u_n = 2^n - 1 \) (see \[1\]) and the main result of this paper still holds provided that \( b \) is not a power of 2. We give no further details.
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