Journal of Computational and Applied Mathematics 237 (2013) 126–135

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

The inverse of banded matrices

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ARTICLE INFO

Article history:
Received 8 June 2010
Received in revised form 8 September 2011

Keywords:
Triangular matrix
Hessenberg matrix
Inverse
r-banded matrix

ABSTRACT

The inverses of r-banded matrices, for $r = 1, 2, 3$ have been thoroughly investigated as one can see from the references we provide. Let $B_{r,n}$ $(1 \le r \le n)$ be an $n \times n$ matrix of entries $\{a_j^i\}$, $-r \le i \le r$, $1 \le j \le r$, with the remaining un-indexed entries all zeros. In this paper, generalizing a method of Mallik (1999) [5], we give the LU factorization and the inverse of the matrix $B_{r,n}$ (if it exists). Our results are valid for an arbitrary square matrix (taking $r = n$), and so, we will give a new approach for computing the inverse of an invertible square matrix. Our method is based on Hessenberg submatrices associated to $B_{r,n}$.

1. Background

There are various methods for finding inverses of matrices (if these exist), and we recall the Gauss–Jordan method, the triangular decomposition such as LUD or Cholesky factorization, to mention only a few. A very popular approach is based on block partitioning. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ whose inverse (called Schur's complement) is

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}B^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}B^{-1} \\ -B^{-1}A_{21}A_{11}^{-1} & B^{-1} \end{bmatrix},$$

where $B = A_{22} - A_{21}A_{11}^{-1}A_{12}$ (under the "right" conditions, namely, that A_{11}, B are invertible). Further, an iterative approach for finding the inverse exists, but its convergence is a function of the matrix's condition number, $cond(A) = \|A\|\|A^{-1}\|$ (in some norm, say, ℓ_∞, where $\|A\|_\infty = \max \sum_j |a_{ij}|$; or ℓ_1, with $\|A\|_1 = \|A^T\|_\infty$). We will not go into details regarding these parameters of a matrix as the considerable literature has been dedicated to these concepts.

Certainly, the inverse of a diagonal matrix (if it exists) is found easily: if $A = diag(a_1, \ldots, a_n)$, then $A^{-1} = diag(a_1^{-1}, \ldots, a_n^{-1})$. If U is a bidiagonal, say $U = \begin{bmatrix} u_1 & c_1 & 0 & \cdots & 0 \\ 0 & u_2 & c_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & u_n \end{bmatrix}$, $u_i \ne 0$, then $U^{-1} = (v_{ij})_{i,j}$, where the entries satisfy the recurrences

$$v_{ij} = \begin{cases} 0, & i > j; \\ \dfrac{1}{u_i}, & i = j \\ -\dfrac{c_i v_{i+1,j}}{u_i}, & i < j. \end{cases}$$

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doi:10.1016/j.cam.2012.07.018

These recurrences can certainly be solved, and we obtain

$$v_{ij} = \begin{cases} 0, & i > j \\ \dfrac{1}{v_j} \displaystyle\prod_{k=1}^{j-1} \left(\dfrac{-c_k}{u_k} \right), & i \le j. \end{cases}$$

Going further to tridiagonal matrices, things start to change. We want to mention here the work of Schlegel [1], who showed that if $B = \begin{bmatrix} b_1 & 1 & 0 & \cdots & 0 \\ 1 & b_2 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 1 & b_n \end{bmatrix}$, then the inverse $B^{-1} = (c_{ij})_{i,j}$ is given by $c_{ij} = r^{-1} r_{i-1} r_{n-i}$, $i \le j$ and c_{ji}, if $j < i$, where $r_0 = 1, r_1 = -b_1, r_k = -(b_k r_{k-1} + r_{k-2}), k = 2, \ldots, n-1$ and $r = b_n r_{n-1} + r_{n-2} = (-1)^{n+1} \det(B)$. We would like to make an observation at this point: since the inverse of a tridiagonal matrix is a full matrix, the Schur's complement method is not very efficient.

Moreover, Vimuri [2] obtained the inverse of another particular tridiagonal matrix in terms of the Gegenbauer polynomial $C_n^\alpha(x)$ (for $\alpha = 1$) whose generating function is $\frac{1}{(1-2xt+t^2)^\alpha} = \sum_{n=0}^{\infty} C_n^\alpha(x) t^n$; Prabhakar et al. [3] showed some connection between the aforementioned inverse and the generalized Hermite polynomials, namely, $g_n^m(x, \lambda) = \sum_{k=0}^{[n/m]} \frac{n!}{k!(n-mk)!} \lambda^k x^{n-mk}$. (Classical Hermite $H_n(x)$ are obtained for $m = 2, \lambda = -1, x \to 2x$.)

Many special cases of the banded matrices such as Toeplitz matrices, symmetric Toeplitz matrices, especially tridiagonal matrices, etc., have been studied by several authors, as we previously mentioned (the reader can find many more references published in the present journal, for instance [4], among others). These matrices arise in many areas of mathematics and its applications. Tridiagonal, or more general, banded matrices are used in telecommunication system analysis, finite difference methods for solving PDEs, linear recurrence systems with non-constant coefficients, etc., so, it is natural to ask the question of whether one can obtain some results about the inverse of 4-diagonal, or perhaps, even general banded matrices. We will do so in this paper.

Throughout this paper, we consider a general r-banded matrix $B_{r,n}$ of order n defined by

$$B_{r,n} = (b_{ij}) = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 & \ldots & a_1^r & 0 & \ldots & 0 \\ a_1^{-2} & a_2^1 & a_2^2 & a_2^3 & \ldots & a_2^r & \ddots & \vdots \\ a_1^{-3} & a_2^{-2} & a_3^1 & a_3^2 & a_3^3 & \ldots & \ddots & 0 \\ \vdots & a_2^{-3} & a_3^{-2} & a_4^1 & a_4^2 & \ddots & \ddots & a_{n-r+1}^r \\ a_1^{-r} & \vdots & a_3^{-3} & a_4^{-2} & a_5^1 & \ddots & a_{n-3}^3 & \vdots \\ 0 & a_2^{-r} & \ldots & \ddots & \ddots & \ddots & a_{n-2}^2 & a_{n-2}^3 \\ \vdots & \ddots & \ddots & \ldots & a_{n-1}^{-3} & a_{n-2}^{-2} & a_{n-1}^1 & a_{n-1}^2 \\ 0 & \ldots & 0 & a_{n-r+1}^{-r} & \ldots & a_{n-2}^{-3} & a_{n-1}^{-2} & a_n^1 \end{bmatrix} \tag{1}$$

where a_j^i's stand for arbitrary real numbers, $-r \le i \le r$, $1 \le j \le r \le n$ (note that the i in the notation a_n^i does not denote the ith power of a_n). When $r = n$, the matrix $B_{n,n}$ is reduced to an arbitrary square matrix.

In this paper, generalizing a method of [5], we give the *LU* factorization and (in our main result) the inverse of the matrix $B_{r,n}$ (if it exists). Our results are valid for an arbitrary square matrix (by taking $r = n$). Therefore we give a new approach for computing the inverse of an invertible square matrix, thus, generalizing various results (see [6–8,5,9,10,1], and the references therein). Our method is based on Hessenberg submatrices associated to $B_{r,n}$.

Section 2 deals with the *LU* factorization of an r-banded matrix. To find the inverse of such a matrix (obtained in Section 4), we need to find inverses of the obtained triangular matrices from the *LU* factorization and this is done in Section 3. We conclude the paper with a few examples in Section 5.

2. *LU* factorization of an *r*-banded matrix

This section is mainly devoted to the *LU* factorization of the matrix $B_{r,n}$. First, we construct two recurrences. For $1 \le i \le r$ and $s \ge r \ge 1$, define

$$k_s^i = a_s^i - \sum_{t=1}^{r-i} m_{s-t}^t k_{s-t}^{t+i} \tag{2}$$

and for $1 \le i \le r-1$

$$m_s^i = \frac{a_s^{-(i+1)} - \sum_{t=1}^{r-i-1} m_{s-t}^{i+t} k_{s-t}^{t+1}}{k_s^1}, \tag{3}$$

with initial conditions

$$m_1^i = \frac{a_1^{-(i+1)}}{k_1^1} = \frac{a_1^{-(i+1)}}{a_1^1}, \quad 1 \le i \le r-1$$

and

$$k_1^i = a_1^i, \quad 1 \le i \le r,$$

where the terms $a_n^{\pm i}$ for $1 \le i \le n$ are the entries of $B_{r,n}$. For these sequences to be well-defined, we assume that none of the denominators k_s^i are zero (which is equivalent to the below-defined U, and consequently, $B_{r,n}$ being invertible).

Now define unitary lower tridiagonal matrix L and upper triangular matrix U,

$$L = (l_{ij}) = \begin{bmatrix} 1 & & & & & & & 0 \\ m_1^1 & 1 & & & & & & \\ m_1^2 & m_2^1 & 1 & & & & & \\ \vdots & m_2^2 & m_3^1 & 1 & & & & \\ m_1^{r-1} & \vdots & m_3^2 & m_4^1 & \ddots & & & \\ 0 & m_2^{r-1} & \ddots & m_4^2 & \ddots & 1 & & \\ \vdots & \ddots & \ddots & \dots & \ddots & m_{n-2}^1 & 1 & \\ 0 & \dots & 0 & m_{n-r+1}^{r-1} & \dots & m_{n-2}^2 & m_{n-1}^1 & 1 \end{bmatrix} \tag{4}$$

and

$$U = (u_{ij}) = \begin{bmatrix} k_1^1 & k_1^2 & k_1^3 & \dots & k_1^r & 0 & \dots & 0 \\ & k_2^1 & k_2^2 & k_2^3 & \dots & k_2^r & & \vdots \\ & & k_3^1 & k_3^2 & k_3^3 & \dots & \ddots & 0 \\ & & & k_4^1 & k_4^2 & \ddots & \dots & k_{n-r+1}^r \\ & & & & k_5^1 & \ddots & k_{n-3}^3 & \vdots \\ & & & & & \ddots & k_{n-2}^2 & k_{n-2}^3 \\ & & & & & & k_{n-1}^1 & k_{n-1}^2 \\ 0 & & & & & & & k_n^1 \end{bmatrix}. \tag{5}$$

Our first result gives the LU factorization of $B_{r,n}$.

Theorem 1. *For $n > 1$, the LU factorization of matrix $B_{r,n}$ is given by*

$$B_{r,n} = LU$$

where L and U are defined as in (4) and (5), respectively.

Proof. First, we consider the case $1 \le i = j \le r$. From matrix multiplication and the definitions of L and U, we have

$$\begin{aligned} b_{ii} &= \sum_{s=1}^{n} l_{i,s} u_{s,i} = \sum_{s=1}^{i} l_{i,s} u_{s,i} \\ &= l_{i,1} u_{1,i} + l_{i,2} u_{2,i} + \cdots + l_{ii} u_{ii} \\ &= m_1^{i-1} k_1^i + m_2^{i-2} k_2^{i-1} + \cdots + m_{i-1}^1 k_{i-1}^2 + k_i^1. \end{aligned}$$

From (2), by taking $r = i$, we obtain

$$b_{ii} = a_i^1,$$

which gives the conclusion. Now consider the case of $i > r$. Thus

$$b_{ii} = \sum_{s=1}^{n} l_{is}u_{si} = \sum_{s=1}^{r} l_{i,i-s+1}u_{i-s+1,i}$$
$$= l_{ii}u_{ii} + l_{i,i-1}u_{i-1,i} + \cdots + l_{i,i-r+1}u_{i-r+1,i}.$$

From the definitions of L and U, we write

$$b_{ii} = k_i^1 + m_{i-1}^1 k_{i-1}^2 + m_{i-2}^2 k_{i-2}^3 + \cdots + m_{i-r+1}^{r-1} k_{i-r+1}^r$$

which, by taking $i = 1$ in (2) implies

$$b_{ii} = a_n^1,$$

which shows the claim for $i = j$.

Next, we look at the super-diagonal entries of matrix $B_{r,n}$. Consider the case $j = i + q$ where $1 \le q \le r - 1$. Thus, by the definition of B_n, for $1 \le q \le r - 1$,

$$b_{i,i+q} = a_i^{1+q} = \sum_{s=1}^{n} l_{i,s}u_{s,i+q}.$$

We consider two cases. First, we assume $1 \le i \le r - q$, and so,

$$\sum_{t=1}^{i} l_{i,t}u_{t,i+q} = l_{i,1}u_{1,i+q} + l_{i,2}u_{2,i+q} + \cdots + l_{ii}u_{i,i+q}$$
$$= m_1^{i-1}k_1^{i+q} + m_2^{i+q}k_2^{i+q-1} + \cdots + m_{i-1}^1 k_{i-1}^{q+1} + k_i^{q+1}$$

which, by taking $i \to q + 1$ and $n \to i$ in (2), gives us

$$\sum_{t=1}^{i} l_{i,t}u_{t,i+q} = a_i^{1+q},$$

which completes the proof for the first case.

Now we consider the case $r - q < i$. Thus we write

$$b_{i,i+q} = a_i^{1+q} = \sum_{s=1}^{n} l_{i,s}u_{s,i+q} = \sum_{t=1}^{r-q} l_{i,i-t+1}u_{i-t+1,i+q}.$$

From the definitions of matrices U and L, we can write

$$\sum_{t=1}^{r-q} l_{i,i-t+1}u_{i-t+1,i+q} = l_{ii}u_{i,i+q} + l_{i,i-1}u_{i-1,i+q} + \cdots + l_{i,i-r-q+1}u_{i-r+q+1,i+q}$$
$$= k_i^{1+q} + m_{i-1}^{i}k_{i-1}^{2+q} + m_{i-2}^2 k_{i-2}^{3+q} + \cdots + m_{i-r+q+1}^{r-q-1}k_{i-r+q+1}^{r}$$
$$= k_i^{1+q} + \sum_{t=1}^{r-q-1} m_{i-t}^{t} k_{i-t}^{q+1+t}.$$

Using (2), we obtain

$$b_{i,i+q} = a_i^{1+q},$$

which completes the proof for the upper diagonal entries of matrix $B_{r,n}$.

Finally, we look at the upper diagonal entries of the matrix $B_{r,n}$, and we need to show that $b_{i+q,i} = a_i^{-(q+1)}$. Here, we first consider the case $1 \le i \le r - q$. From the definitions of matrices U and L,

$$b_{i+q,i} = \sum_{t=1}^{n} l_{i+q,t}u_{t,i} = \sum_{t=1}^{i} l_{i+q,t}u_{t,i}$$
$$= m_1^{i+q-1}k_1^{i} + m_2^{i+q-2}k_2^{-1} + \cdots + m_{i-1}^{q+1}k_{i-1}^{2} + m_i^{q}k_i^{1}$$
$$= m_i^{q}k_i^{1} + \sum_{t=1}^{i-1} m_{i-t}^{q+t}k_{i-t}^{t+1}$$

which, by taking $n \to i$ and $i \to q$ in (2), implies

$$b_{i+q,i} = a_i^{-(q+1)}.$$

For the final case, that is, $i > r - q$, using the definitions of U_1 and L_1, we write

$$\begin{aligned} b_{i+q,i} &= \sum_{t=1}^{n} l_{i+q,t} u_{t,i} = \sum_{t=1}^{r-q} l_{i+q,i-t+1} u_{i-t+1,i} \\ &= l_{i+q,i} u_{i,i} + l_{i+q,i-1} u_{i-1,i} + \cdots + l_{i+q,i-r+q+1} u_{i-r+q+1,i} \\ &= m_i^q k_i^1 + m_{i-1}^{q+1} k_{i-1}^2 + \cdots + m_{i-r+q+1}^{q+(r-q-1)} k_{i-r+q+1}^{r-q} \\ &= m_i^q k_i^1 + \sum_{t=1}^{r-q-1} m_{i-t}^{q+t} k_{i-t}^{t+1} \end{aligned}$$

which, by taking $n \to i$ and $i \to q$ in (2), gives

$$b_{i+q,i} = a_i^{-(q+1)},$$

and the theorem is proved. □

The result of Theorem 1 will be valid for the LU factorization of any arbitrary square matrix by taking $r = n$ in the matrix $B_{r,n}$.

Now we give a closed formula for $\det \left(B_{r,n}\right)$ using the LU factorization of $B_{r,n}$.

Corollary 2. *For $n > 0$,*

$$\det B_{r,n} = \prod_{i=1}^{n} k_i^1$$

where k_i^1's are given by (2).

3. The inverse of triangular matrices

In this section we give an explicit formula for the inverse of a general triangular matrix. For this purpose, we construct certain submatrices of a triangular matrix, which are Hessenberg matrices, and then we consider the determinants of these submatrices to determine the entries of the inverse of the considered triangular matrix. Since upper and lower triangular matrices have, essentially, the same properties, first we consider the upper triangular matrix case. We denote the corresponding Hessenberg matrices for an upper and a lower triangular matrix by $H_u(r,s)$ and $H_\ell(r,s)$, respectively.

Let $H = \left(h_{ij}\right)$ be an arbitrary $(n \times n)$ upper triangular matrix. Now we construct square Hessenberg submatrices of H of order $|s-r|$ in the following way: for $s > r > 0$, let $H_u(r,s) = \left(\tilde{h}_{ij}\right)$ denote an upper Hessenberg submatrix of H by deleting its first r and last $(n-s)$ columns, and, first $(r-1)$ and last $(n-s+1)$ rows. Clearly the $(s-r) \times (s-r)$ upper Hessenberg matrix $H_u(r,s)$ takes the form:

$$H_u(r,s) = \begin{bmatrix} h_{r,r+1} & h_{r,r+2} & \cdots & h_{r,s-1} & h_{r,s} \\ h_{r+1,r+1} & h_{r+1,r+2} & \cdots & h_{r+1,s-1} & h_{r+1,s} \\ 0 & \ddots & & \vdots & \vdots \\ \vdots & \cdots & h_{s-2,s-2} & h_{s-2,s-1} & h_{s-2,s} \\ 0 & \cdots & 0 & h_{s-1,s-1} & h_{s-1,s} \end{bmatrix}. \tag{6}$$

Similarly let $H = \left(h_{ij}\right)$ be an arbitrary $(n \times n)$ lower triangular matrix. We construct square Hessenberg submatrices of H of order $(r-s)$ in the following way: for $r > s > 0$, let $H_\ell(r,s) = \left(\hat{h}_{ij}\right)$ denote a lower Hessenberg submatrix of H by deleting its first r and last $(n-s)$ rows, and, first $(r-1)$ and last $(n-s+1)$ columns. Clearly the $(r-s) \times (r-s)$ lower Hessenberg matrix $H_\ell(r,s)$ takes the form:

$$H_\ell(r,s) = \begin{bmatrix} h_{r+1,r} & h_{r+1,r+1} & 0 & & 0 \\ h_{r+2,r} & h_{r+2,r+1} & \ddots & & \\ \vdots & \vdots & \ddots & h_{s-2,s-2} & \\ h_{s-1,r} & h_{s-1,r+1} & \cdots & h_{s-1,s-2} & h_{s-1,s-1} \\ h_{s,r} & h_{s,r+1} & \cdots & h_{s,s-2} & h_{s,s-1} \end{bmatrix}.$$

Here we note that $(H_u(r,s))^T = (H_\ell(s,r))$. Our construction is valid for any upper or lower triangular matrix, but we will apply it only to L, U, given by (4) and (5), rendering $L_\ell(i,j)$, $U_u(i,j)$.

For example, let A be an upper triangular matrix of order 6 as follows:

$$A = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ 0 & a_2 & b_2 & c_2 & d_2 & e_2 \\ 0 & 0 & a_3 & b_3 & c_3 & d_3 \\ 0 & 0 & 0 & a_4 & b_4 & c_4 \\ 0 & 0 & 0 & 0 & a_5 & b_5 \\ 0 & 0 & 0 & 0 & 0 & a_6 \end{bmatrix}.$$

Thus $A_u(2,5)$ and $A_u(3,6)$ take the forms :

$$A_u(2,5) = \begin{bmatrix} b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 \\ 0 & a_4 & b_4 \end{bmatrix} \quad \text{and} \quad A_u(3,6) = \begin{bmatrix} b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 \\ 0 & a_5 & b_5 \end{bmatrix}.$$

Here note that all matrices of the form $H_u(i,j)$ $(H_\ell(i,j))$ obtained from an upper (lower) triangular matrix U are upper (lower) Hessenberg matrices.

Now we start with the following two lemmas. Throughout this paper, we assume the boundary conditions $H_u(r,r) = H_\ell(r,r) = 1$ and $\prod_i^j x_i = 1$ for $i > j$.

Lemma 3. *Let the $(j-i)\times(j-i)$ upper Hessenberg matrix $H_u(i,j)$ be defined as in (6). Then, for $j > i+1$,*

$$\det H_u(i,j) = \sum_{t=1}^{j-i} \left\{ (-1)^{2j-t} a_{j-t,j} \det(H_u(i,j-t)) \prod_{k=j-1-t}^{j-1} a_{kk} \right\}.$$

Proof. If we compute the determinant of the upper Hessenberg matrix $H_u(i,j)$ by the Laplace expansion of a determinant with respect to the last column, then the proof follows. □

Theorem 4. *Let $U = (a_{ij})$ be an $(n \times n)$ arbitrary upper triangular matrix and $W = (w_{ij}) = U^{-1}$, its inverse. Then*

$$w_{ij} = \begin{cases} (a_{ii})^{-1} & \text{if } i = j, \\ \left(\prod_{k=i}^{j} a_{kk} \right)^{-1} (-1)^{i+j} \det(H_u(i,j)) & j > i, \end{cases}$$

where $H_u(r,s)$ is as before.

Proof. Denote WU by $E = (e_{ij})$. It is clear that for the case $i = j$, $E = I_n$ where I_n is the nth unit matrix. Now consider the case $j > i$. From the definitions of matrices W and U,

$$\begin{aligned} e_{ij} &= \sum_{t=1}^{n} w_{it} a_{tj} = \sum_{t=i}^{j} w_{it} a_{tj} \\ &= \frac{a_{ij}}{a_{ii}} + \sum_{t=i+1}^{j} \left(\prod_{k=i}^{t} a_{kk} \right)^{-1} (-1)^{i+t} a_{tj} \det(H_u(i,t)) \\ &= \frac{a_{ij}}{a_{ii}} + \frac{(-1)^{2i+1} a_{i+1,j} \det(H_u(i,i+1))}{a_{ii} a_{i+1,i+1}} + \frac{(-1)^{2i+2} a_{i+2,j} \det(H_u(i,i+2))}{a_{ii} a_{i+1,i+1} a_{i+2,i+2}} \\ &\quad + \cdots + \frac{(-1)^{i+j-2} a_{j-2,j} \det(H_u(i,j-2))}{\prod_{k=i}^{j-2} a_{kk}} + \frac{(-1)^{i+j-1} a_{j-1,j} \det(H_u(i,j-1))}{\prod_{k=i}^{j-1} a_{kk}} + \frac{(-1)^{i+j} \det(H_u(i,j))}{\prod_{k=i}^{j-1} a_{kk}} \\ &= \left(\prod_{k=i}^{j-1} a_{kk} \right)^{-1} \left(a_{ij} (-1)^{2i} \prod_{k=i+1}^{j-1} a_{kk} + a_{i+1,j} (-1)^{2i+1} \prod_{k=i+2}^{j-1} a_{kk} \det(H_u(i,i+1)) \right. \\ &\quad + \cdots + a_{j-2,j} (-1)^{i+j-2} \prod_{k=j-1}^{j-1} \det(H_u(i,j-2)) + (-1)^{i+j-1} a_{j-1,j} \det(H_u(i,j-1)) \\ &\quad \left. + (-1)^{i+j} \det(H_u(i,j)) \right) \end{aligned}$$

$$= \left(\prod_{k=i}^{j-1} a_{kk}\right)^{-1} \left\{\sum_{t=1}^{j-i}\left(\left(\prod_{k=i+t}^{j-1} a_{kk}\right)(-1)^{2i+t-1} a_{i+t-1,j} \det\left(H_u\left(i, i+t-1\right)\right)\right) + (-1)^{i+j} \det\left(H_u\left(i,j\right)\right)\right\},$$

which, by Lemma 3, implies $e_{ij} = 0$, and the proof is completed. □

All the results of this section hold for lower triangular matrices, with the obvious modification using the lower Hessenberg matrix.

Now we mention an interesting fact that the numbers of summed or subtracted terms in computing the inverse of a term of an upper (lower) triangular matrix are the generalized order-k Fibonacci numbers defined by

$$f_n^k = \sum_{i=1}^{k} c_i f_{n-i}^k, \quad n > 0,$$

where $f_{1-k} = 1, f_{-k}^k = \cdots = f_0^k = 0$.

When $k = 2$ and $c_1 = c_2 = 1$, the generalized order-2 Fibonacci numbers are the usual Fibonacci numbers, that is, $f_m^2 = F_m$ (mth Fibonacci number). When also $k = 3$, $c_1 = c_2 = c_3 = 1$, then the generalized order-3 Fibonacci numbers by the initial conditions $f_{-2}^3 = 1, f_0^3 = f_{-1}^3 = 0$, are

$$1, 1, 2, 4, 7, 13, 24, \ldots,$$

which are also known as *tribonacci* numbers.

Let $U_n = \left(u_{ij}\right)$ be an upper (with k super-diagonals) triangular matrix of order n with $u_{ii} = h_i$ for $1 \le i \le n$, $u_{i,i+r} = \sum_{t=1}^{c_r} a_{r,i}$ for $1 \le i \le n-r$, $1 \le r \le k$ and h_i's are all distinct from zero, and, $a_{r,i}$' s are arbitrary.

For computing the inverse of U_n, we need the corresponding Hessenberg submatrices defined as before by $H\left(r, s\right)$. Therefore we should note that the numbers of summed or subtracted terms in computing the inverse of a term of U_n omitting the signs and denominators of the terms, that is, the number of required summations in the expansion of $\det\left(H\left(r, s\right)\right)$, are the generalized order-$(s-r)$ Fibonacci numbers, f_n^{s-r}. To show that, we consider a $(k \times k)$ upper Hessenberg matrix $H_k = \left(a_{ij}\right)$ with r-superdiagonals whose entries are given by $a_{i+1,i} = e_i$ for $1 \le i \le k-1$, $a_{i,i+r} = \sum_{t=1}^{c_r} h_{t,i}^{(r)}$ for $1 \le i \le k-r$ and $0 \le r \le k$, $e_i \ne 0$ for all i. If one superdiagonal has a 0 entry, then all the entries in this superdiagonal are zeros. That is, if $a_{i,i+r} = 0$ for some i and r, then $c_r = 0$.

Here E_n denotes the number of summed or subtracted terms in $\det H_n$. For example, if

$$H_3 = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & g & h \end{bmatrix},$$

then $\det H_3 = ahe - afg - bdh + cdg$ and so, the corresponding $E_3 = 4$.

By expanding $\det H_n$ with respect to the first row without any simplification in the entries $h_i^{(r)} = \sum_{t=1}^{c_r} a_{r,i}$, we get

$$E_n = c_1 E_{n-1} + c_2 E_{n-2} + \cdots + c_r E_{n-r}.$$

One easily computes that $E_1 = 1, E_2 = 2, E_3 = 4, \ldots, E_r = 2^{r-1}$.

Consequently one can see that $E_n = f_{n+1}^k$ where f_n^k is the generalized order-k Fibonacci numbers.

For example, for $k = 4$, $r = 2$, let $c_1 = 2$ and $c_2 = 1$, that is, $a_{i,i+1} = \sum_{t=1}^{2} h_{t,i}^{(1)} = h_{1,i}^{(1)} + h_{2,i}^{(1)}$, $a_{i,i+2} = \sum_{t=1}^{1} h_{t,i}^{(2)} = h_{1,i}^{(2)}$ and so

$$H_4 = \begin{bmatrix} h_{1,1}^{(1)} + h_{2,1}^{(1)} & h_{1,1}^{(2)} & 0 & 0 \\ e_1 & h_{1,2}^{(1)} + h_{2,2}^{(1)} & h_{1,2}^{(2)} & 0 \\ 0 & e_2 & h_{1,3}^{(1)} + h_{2,3}^{(1)} & h_{1,3}^{(2)} \\ 0 & 0 & e_3 & h_{1,4}^{(1)} + h_{2,4}^{(1)} \end{bmatrix}.$$

Since $r = 2$, $c_1 = 2$, $c_2 = 1$, the counting sequence E_n satisfies

$$E_n = 2E_{n-1} + E_{n-2},$$

with $E_1 = 1$, $E_2 = 2$, so $E_n = P_n$, the well known Pell sequence.

Therefore, the number of summed or subtracted terms while computing $\det H_4$ is the 5th Pell number:

$$\det \begin{bmatrix} h_{1,1}^{(1)} + h_{2,1}^{(1)} & h_{1,1}^{(2)} & 0 & 0 \\ e_1 & h_{1,2}^{(1)} + h_{2,2}^{(1)} & h_{1,2}^{(2)} & 0 \\ 0 & e_2 & h_{1,3}^{(1)} + h_{2,3}^{(1)} & h_{1,3}^{(2)} \\ 0 & 0 & e_3 & h_{1,4}^{(1)} + h_{2,4}^{(1)} \end{bmatrix}$$

$$= e_1 e_3 h_{1,1}^2 h_{1,3}^2 + h_{1,1} h_{1,2} h_{1,3} h_{1,4} + h_{1,1} h_{1,2} h_{1,3} h_{2,4} + h_{1,1} h_{1,2} h_{1,4} h_{2,3} + h_{1,1} h_{1,3} h_{2,2} h_{1,4}$$

$$+h_{1,2}h_{2,1}h_{1,3}h_{1,4}+h_{1,1}h_{1,2}h_{2,3}h_{2,4}+h_{1,1}h_{1,3}h_{2,2}h_{2,4}+h_{1,1}h_{2,2}h_{1,4}h_{2,3}+h_{1,2}h_{2,1}h_{1,3}h_{2,4}$$
$$+h_{1,2}h_{2,1}h_{1,4}h_{2,3}+h_{2,1}h_{1,3}h_{2,2}h_{1,4}+h_{1,1}h_{2,2}h_{2,3}h_{2,4}+h_{1,2}h_{2,1}h_{2,3}h_{2,4}+h_{2,1}h_{1,3}h_{2,2}h_{2,4}$$
$$+h_{2,1}h_{2,2}h_{1,4}h_{2,3}+h_{2,1}h_{2,2}h_{2,3}h_{2,4}-e_1h_{1,1}^2h_{1,3}h_{1,4}-e_2h_{1,1}h_{1,2}^2h_{1,4}-e_3h_{1,1}h_{1,2}h_{1,3}^2$$
$$-e_1h_{1,1}^2h_{1,3}h_{2,4}-e_1h_{1,1}^2h_{1,4}h_{2,3}-e_2h_{1,2}^2h_{2,1}h_{1,4}-e_2h_{1,1}h_{1,2}^2h_{2,4}-e_3h_{1,1}h_{1,3}^2h_{2,2}$$
$$-e_3h_{1,2}h_{2,1}h_{1,3}^2-e_1h_{1,1}^2h_{2,3}h_{2,4}-e_2h_{1,2}^2h_{2,1}h_{2,4}-e_3h_{2,1}h_{1,3}^2h_{2,2}.$$

From the above example, it is seen that there are 29 terms in the expansion of $\det H_4$ which is the 5th Pell number, P_5.

4. The inverse of an r-banded matrix

In this section, we give a closed formula for the inverse of an r-banded matrix. First, we consider the inverse of the upper triangular matrix L_1. Here we recall a well known fact that the inverse of an upper triangular matrix is also upper triangular. We shall give the following lemmas whose proofs are straightforward.

Lemma 5. *Let the lower triangular matrix L be as in (4). Let $E = (e_{ij})$ denote the inverse of L. Then*

$$e_{ij}=\begin{cases}1, & \text{if } i=j,\\ (-1)^{i+j}\det\left(L_\ell\left(i,j\right)\right), & \text{if } j<i,\end{cases}$$

where $L_\ell\,(i,j)$ is defined as before.

Lemma 6. *Let the upper triangular matrix U be as in (5). Let $G = (g_{ij})$ denote the inverse of U. Then*

$$g_{ij}=\begin{cases}\dfrac{1}{k_i^1}, & \text{if } j=i,\\ \dfrac{(-1)^{i+j}\det\left(U_u\left(i,j\right)\right)}{\prod\limits_{r=i}^{j}k_r^1}, & \text{if } j>i,\end{cases}$$

where $U_u\,(i,j)$ is defined as before.

The main result of this section follows from the LU factorization of $B_{r,n}$ and the previous lemmas.

Theorem 7. *Let $D_n = (d_{ij})$ denote the inverse of the matrix $B_{r,n}$. Then $d_{ij} = \sum_{t=1}^{n} g_{it}e_{tj}$, where g_{it}, e_{tj} are defined in the previous lemmas. Precisely, if we let $S(i,t,j) = \frac{\det(U_u(i,j))\det(L_\ell(t,j))}{\prod_{r=i}^{j}k_r^1}$, then*

$$d_{ij}=\begin{cases}(-1)^{i+j}\dfrac{\det(U_u(i,j))}{\prod\limits_{r=i}^{j}k_r^1}+(-1)^{i+j}\sum\limits_{t>j}^{n}S(i,t,j), & \text{if } i<j\\ \dfrac{1}{k_i^1}+\sum\limits_{t>i}^{n}S(i,t,i), & \text{if } i=j\\ \dfrac{1}{k_i^1}(-1)^{i+j}\det(L_\ell(i,j))+(-1)^{i+j}\sum\limits_{t>i}^{n}S(i,t,j), & \text{if } i>j.\end{cases}$$

Proof. Since $B_{r,n} = LU$, then by the previous two lemmas, we get $D = U^{-1}L^{-1}$, and so,

$$d_{ij}=\sum_{t=1}^{n}g_{it}e_{tj}=\sum_{t=\max\{i,j\}}^{n}g_{it}e_{tj}.$$

By taking the three cases $i<j$, $i=j$, $i>j$ and using the expressions of g_{it}, e_{tj} from Lemmas 5 and 6, the claim follows. □

5. Some particular cases

If we take $r = 2$, and $S_{2,n}$ to be a symmetric matrix, and label $a_i^1 = a_i$, $b_i^2 = b_i^{-2} = -b_{i+1}$, we obtain the formulas of [11] for the inverse of

$$S_{2,n} = \begin{bmatrix} a_1 & -b_2 & \cdots & \cdots & \cdots & \cdots & 0 \\ -b_2 & a_2 & -b_3 & \cdots & \cdots & \cdots & 0 \\ 0 & -b_3 & a_3 & -b_4 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & -b_{n-1} & a_{n-1} & -b_n \\ 0 & \cdots & \cdots & \cdots & 0 & -b_n & a_n \end{bmatrix},$$

namely, the existence of two sequences u_i, v_i such that

$$S_{2,n}^{-1} = \begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 & \cdots & u_1v_n \\ u_1v_2 & u_2v_2 & u_2v_3 & \cdots & u_2v_n \\ u_1v_3 & u_2v_3 & u_3v_3 & \cdots & u_3v_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_1v_n & u_2v_n & u_3v_n & \cdots & u_nv_n \end{bmatrix}.$$

The sequences u_i, v_j can be determined (as Meurant did in [11]) using the LU decomposition of Theorem 1, and they will depend on our k_i^j, m_i^j. We will not repeat the argument here.

Consider the general binary recurrence $G_{n+1} = a\,G_n + b\,G_{n-1}$, $G_0 = 0$, $G_1 = 1$. Let

$$G_{3,n} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -a & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ -b & -a & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & -a & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & -b & -a & 1 \end{bmatrix}.$$

Its inverse is [12]

$$G_{3,n}^{-1} = \begin{bmatrix} G_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ G_2 & G_1 & 0 & \cdots & \cdots & \cdots & 0 \\ G_3 & G_2 & G_1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ G_{n-1} & G_{n-2} & \cdots & \cdots & G_2 & G_1 & 0 \\ G_n & G_{n-1} & \cdots & \cdots & G_3 & G_2 & G_1 \end{bmatrix}.$$

In the same manner, by going through our argument, one can obtain many other known (and possibly unknown) matrix inverses and determinants. We end by displaying, yet another example, based on the Chebyshev polynomials of the second kind, $U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$, $\cos\theta = x$, which satisfy the recurrence: $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$, $U_0(x) = 1$, $U_1(x) = 2x$. The symmetric Toeplitz $S = \begin{bmatrix} a & b & 0 & \cdots & 0 \\ b & a & b & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & b & a & b \\ 0 & \cdots & \cdots & b & a \end{bmatrix}$ has the inverse [13] $S^{-1} = (t_{ij})_{ij}$ where $t_{ij} = \begin{cases} (-1)^{i+j}\frac{1}{b}\,\frac{U_{i-1}(a/2b)U_{n-j}(a/2b)}{U_n(a/2b)}, & i \le j \\ (-1)^{i+j}\frac{1}{b}\,\frac{U_{j-1}(a/2b)U_{n-i}(a/2b)}{U_n(a/2b)}, & i > j. \end{cases}$

It could be interesting to use a variation of the methods of this paper to investigate the spectrum of general r-banded matrices (see [14], for example).

Acknowledgments

The authors thank the referee for pointing out several shortcomings in the presentation of the paper.

The second author was partially supported by Air Force–CVAQ (D. Nussbaum) and NPS-RIP.

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