16.8
Stokes' Theorem

In this section, we will learn about:

The Stokes' Theorem and

using it to evaluate integrals.

STOKES' VS. GREEN'S THEOREM

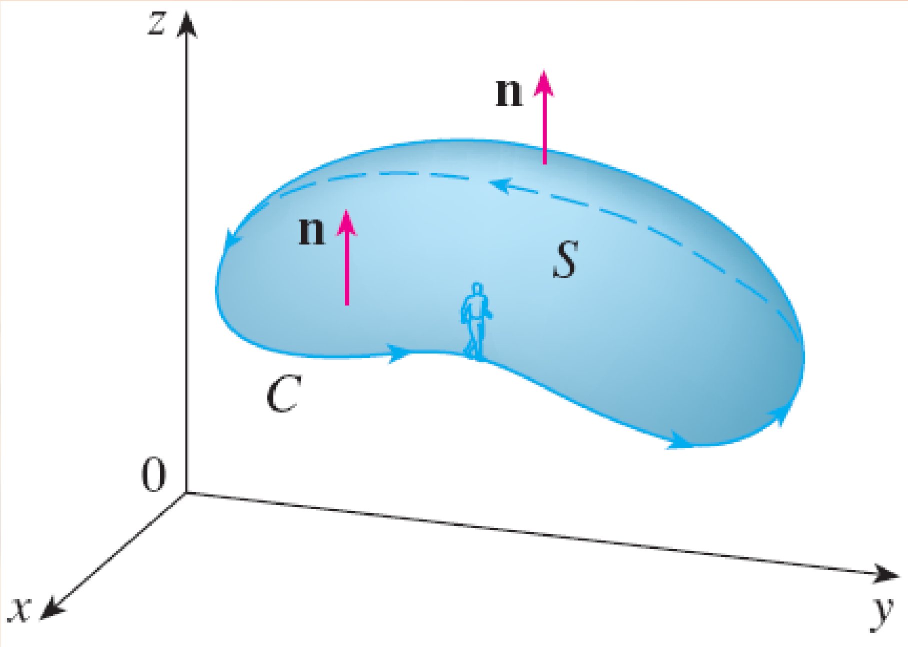
Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem.

- Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve.

- Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (a space curve).

Oriented surface with unit normal vector **n**.

- The orientation of *S* induces the positive orientation of the boundary curve *C*.
- If you walk in the positive direction around *C* with your head pointing in the direction of **n**, the surface will always be on your left.



STOKES' THEOREM

Let:

- *S* be an oriented piecewise-smooth surface bounded by a simple, closed, piecewise-smooth boundary curve *C* with positive orientation.

- **F** be a vector field whose components have continuous partial derivatives on an open region in R^3 that contains *S*.

Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

STOKES' THEOREM

The theorem is named after the Irish mathematical physicist Sir George Stokes (1819–1903).

- What we call Stokes' Theorem was actually discovered by the Scottish physicist Sir William Thomson (1824–1907, known as Lord Kelvin).
- Stokes learned of it in a letter from Thomson in 1850.

STOKES' THEOREM

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

and

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$$

Thus, Stokes' Theorem says:

- The line integral around the boundary curve of *S* of the tangential component of **F** is equal to the surface integral of the normal component of the curl of **F**.

The positively oriented boundary curve of the oriented surface *S* is often written as ∂*S*.

So, the theorem can be expressed as:

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

STOKES' THEOREM, GREEN'S THEOREM, & FTC

There is an analogy among Stokes' Theorem, Green's Theorem, and the Fundamental Theorem of Calculus (FTC).

- As before, there is an integral involving derivatives on the left side of Equation 1 (recall that curl **F** is a sort of derivative of **F**).
- The right side involves the values of **F** only on the boundary of *S*.

In fact, consider the special case where the surface S is flat, in the *xy*-plane with upward orientation.

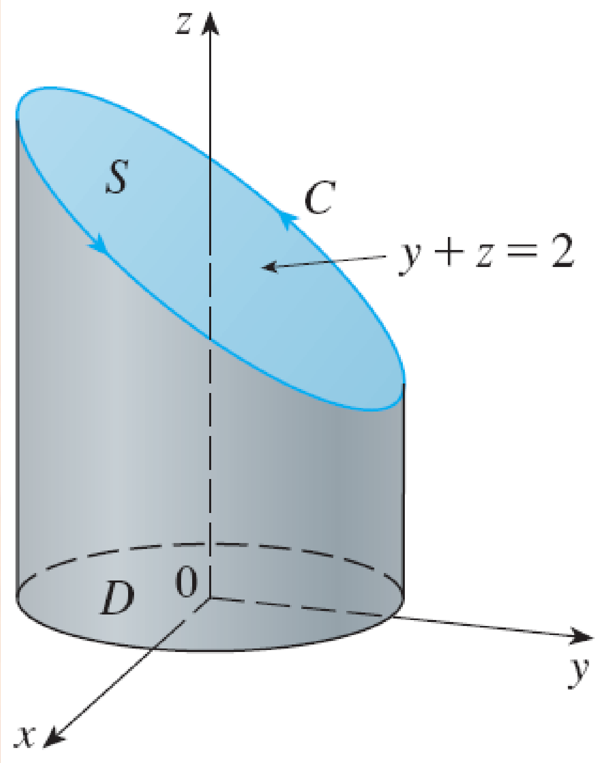
Then:

- The unit normal is **k**.
- The surface integral becomes a double integral.
- Stokes' Theorem becomes:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, dA$$

- Thus, we see that Green's Theorem is really a special case of Stokes' Theorem.

- Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where:
- $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$
- C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.
  (Orient C to be counterclockwise when viewed from above.)
- $\int_C \mathbf{F} \cdot d\mathbf{r}$ could be evaluated directly, however, it's easier to use Stokes' Theorem.

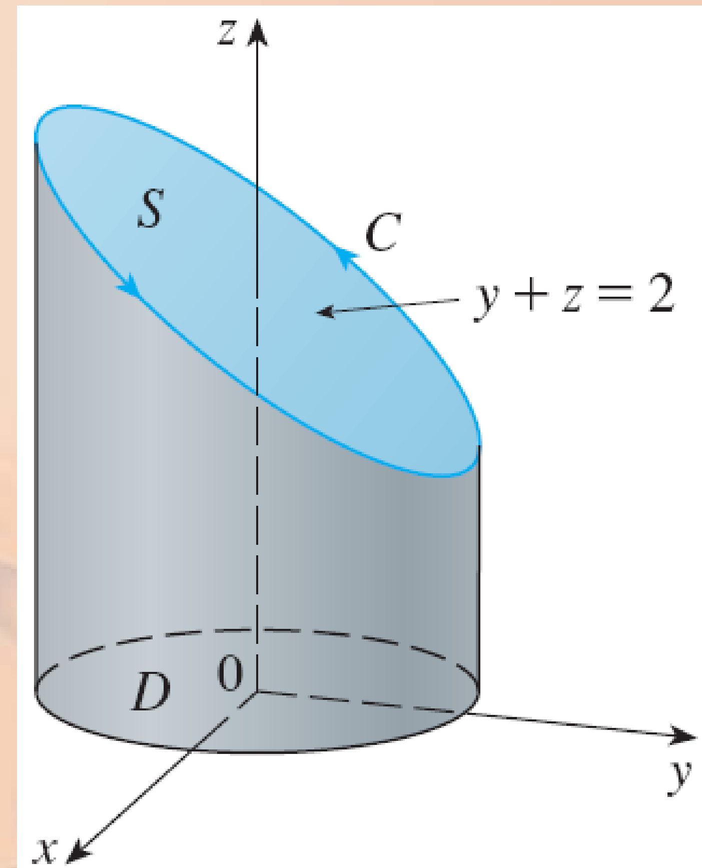


We first compute for $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$:

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1+2y)\,\mathbf{k}$$

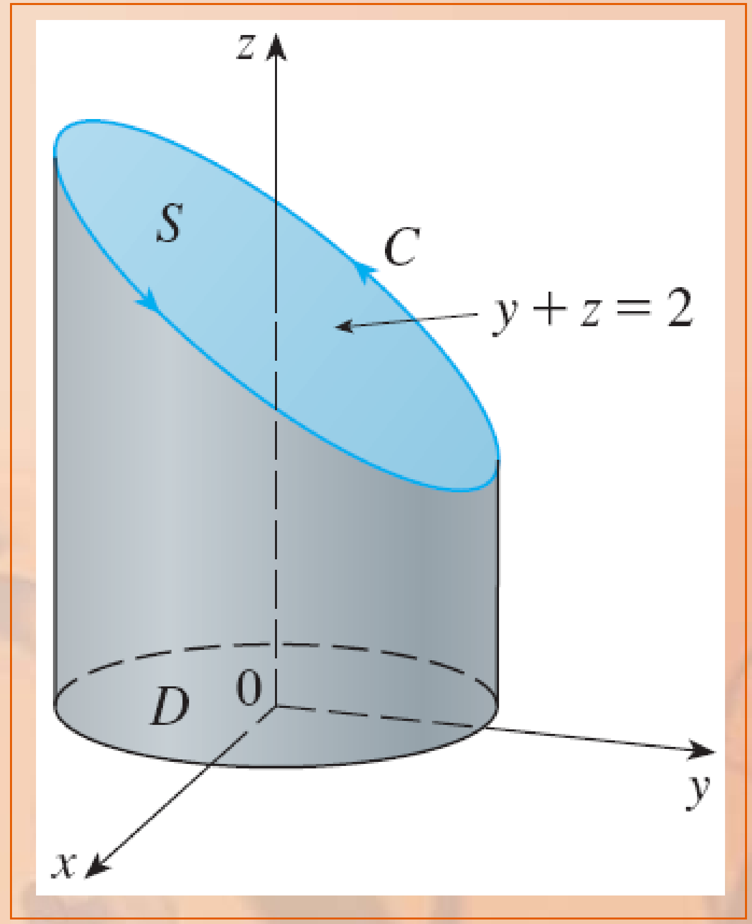
There are many surfaces with boundary *C*.

- The most convenient choice, though, is the elliptical region *S* in the plane $y + z = 2$ that is bounded by *C*.
- If we orient *S* upward, *C* has the induced positive orientation.



The projection D of S on the xy-plane is the disk $x^2 + y^2 \leq 1$.

- So, using Equation 10 in Section 16.7 with $z = g(x, y) = 2 - y$, we have the following result.



$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D (1 + 2y)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r \sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} + 2\frac{r^3}{3} \sin\theta \right]_0^1 d\theta$$

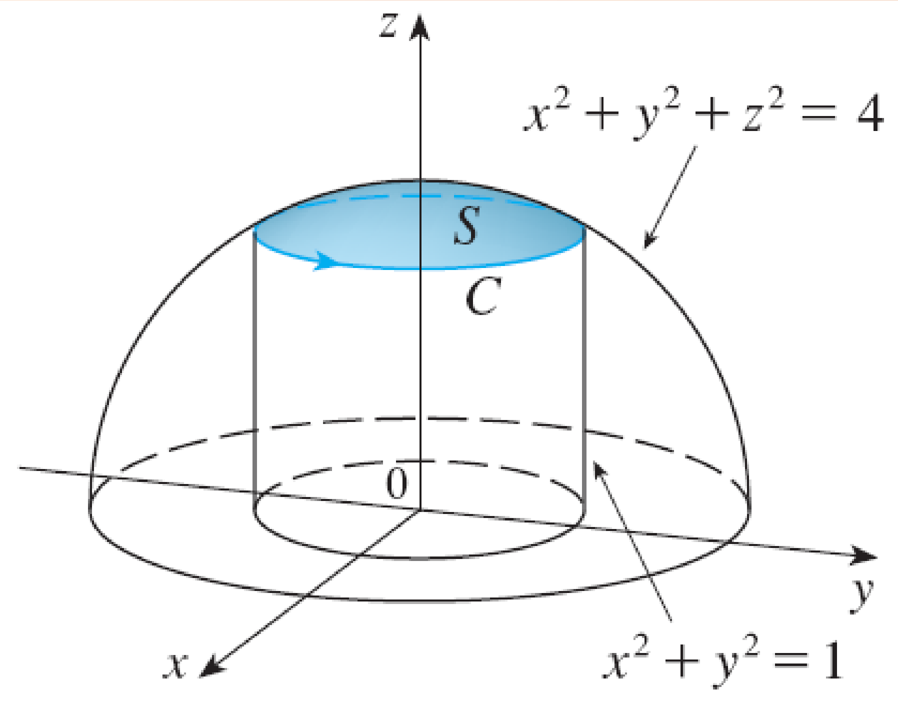
$$= \int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{2}{3} \sin\theta \right) d\theta$$

$$= \tfrac{1}{2}(2\pi) + 0 = \pi$$

Use Stokes' Theorem to compute $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$

where:

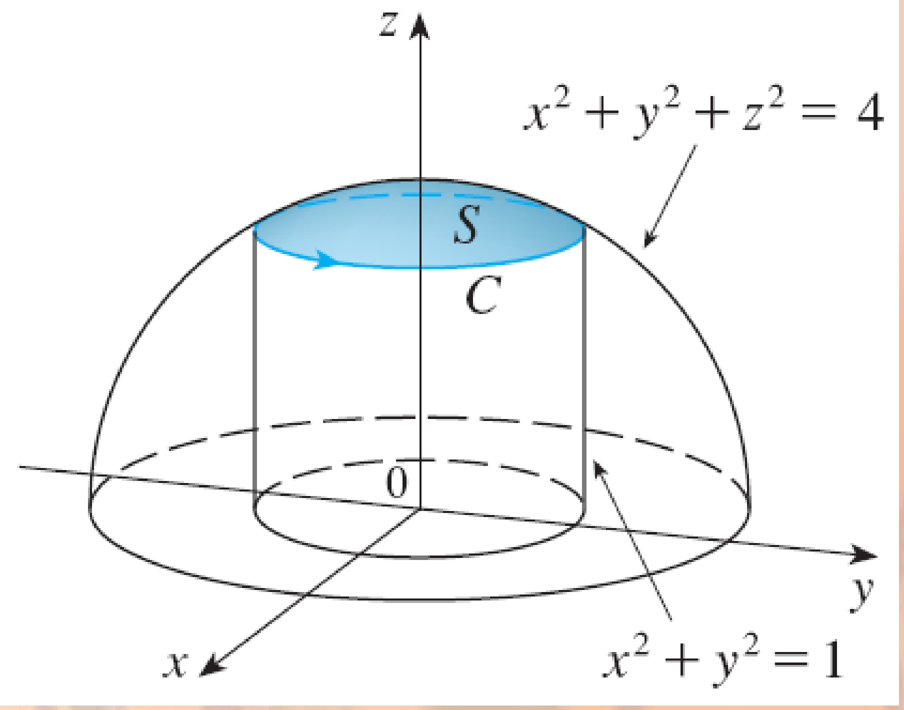
- $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + xy\,\mathbf{k}$
- S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.



To find the boundary curve *C*,

we solve: $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$

- Subtracting, we get $z^2 = 3$, and (since $z > 0$), $z = \sqrt{3}$
- So, *C* is the circle given by: $x^2 + y^2 = 1$, $z = \sqrt{3}$



A vector equation of C is:

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sqrt{3}\,\mathbf{k} \qquad 0 \le t \le 2\pi$$

- Therefore, $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$

Also, we have:

$$\mathbf{F}\left(\mathbf{r}\left(t\right)\right) = \sqrt{3}\cos t\,\mathbf{i} + \sqrt{3}\sin t\,\mathbf{j} + \cos t\sin t\,\mathbf{k}$$

Thus, by Stokes' Theorem,

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} \left(-\sqrt{3} \cos t \sin t + \sqrt{3} \sin t \cos t \right) dt$$

$$= \sqrt{3} \int_0^{2\pi} 0\, dt = 0$$

STOKES' THEOREM

- Note that, in Example 2, we computed a surface integral simply by knowing the values of **F** on the boundary curve *C*.
- This means that:
  - If we have another oriented surface with the same boundary curve *C*, we get exactly the same value for the surface integral!
- In general, if S_1 and S_2 are oriented surfaces with the same oriented boundary curve *C* and both satisfy the hypotheses of Stokes' Theorem, then

$$\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

  - This fact is useful when it is difficult to integrate over one surface but easy to integrate over the other.

CURL VECTOR

We now use Stokes' Theorem to throw some light on the meaning of the curl vector.

- Suppose that *C* is an oriented closed curve and **v** represents the velocity field in fluid flow.

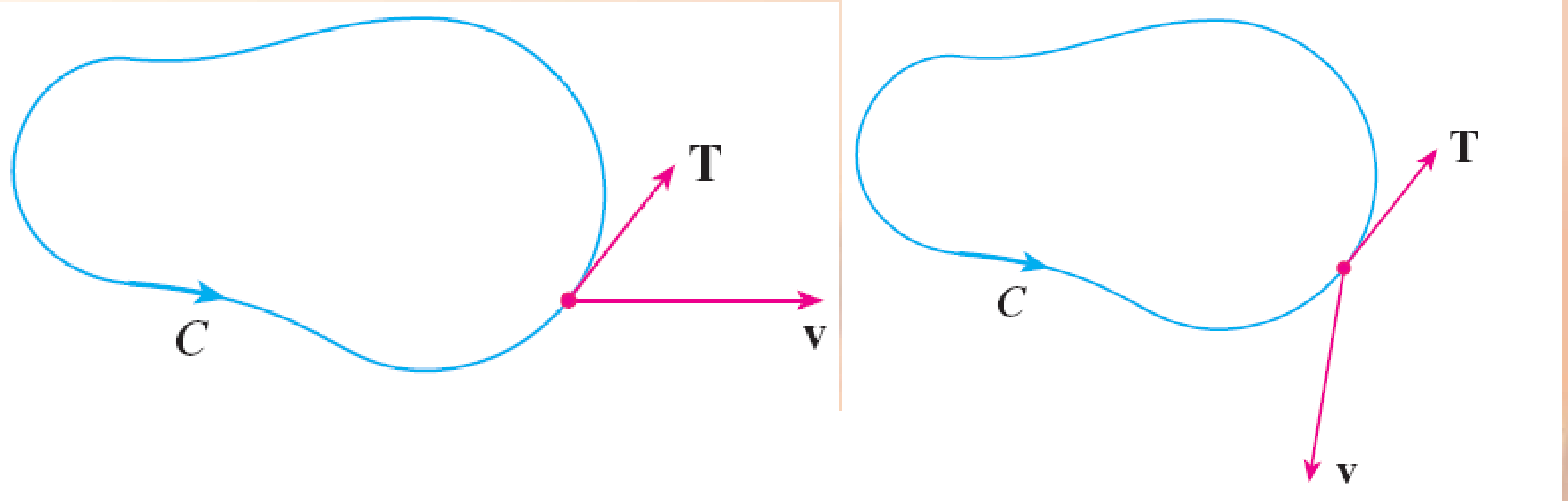
Consider the line integral $\int_C \mathbf{v} \cdot d\mathbf{r} = \int_C \mathbf{v} \cdot \mathbf{T}\, ds$ and recall that **v** · **T** is the component of **v** in the direction of the unit tangent vector **T**.

- This means that the closer the direction of **v** is to the direction of **T**, the larger the value of **v** · **T**.

CIRCULATION

Thus, $\int_C \mathbf{v} \cdot d\mathbf{r}$ is a measure of the tendency of the fluid to move around *C*.

- It is called the circulation of **v** around *C*.



(a) $\int_C \mathbf{v} \cdot d\mathbf{r} > 0$, positive circulation (b) $\int_C \mathbf{v} \cdot d\mathbf{r} < 0$, negative circulation

CURL VECTOR

•Now, let: $P_0(x_0, y_0, z_0)$ be a point in the fluid, and S_a be a small disk with radius a and center P_0.

- Then, (curl **F**)(P) ≈ (curl **F**)(P_0) for all points P on S_a because curl **F** is continuous.
- Thus, by Stokes' Thm., we get the following approximation to the circulation around the boundary circle C_a:

$$\begin{aligned}\int_{C_a} \mathbf{v}\cdot d\mathbf{r} &= \iint_{S_a} \operatorname{curl}\mathbf{v}\cdot d\mathbf{S} = \iint_{S_a} \operatorname{curl}\mathbf{v}\cdot\mathbf{n}\,dS \\ &\approx \iint_{S_a} \operatorname{curl}\mathbf{v}(P_0)\cdot\mathbf{n}(P_0)\,dS \\ &= \operatorname{curl}\mathbf{v}(P_0)\cdot\mathbf{n}(P_0)\,\pi a^2\end{aligned}$$

- The approximation becomes better as $a \to 0$. Thus, we have:

$$\operatorname{curl}\mathbf{v}(P_0)\cdot\mathbf{n}(P_0) = \lim_{a\to 0}\frac{1}{\pi a^2}\int_{C_a}\mathbf{v}\cdot d\mathbf{r}$$

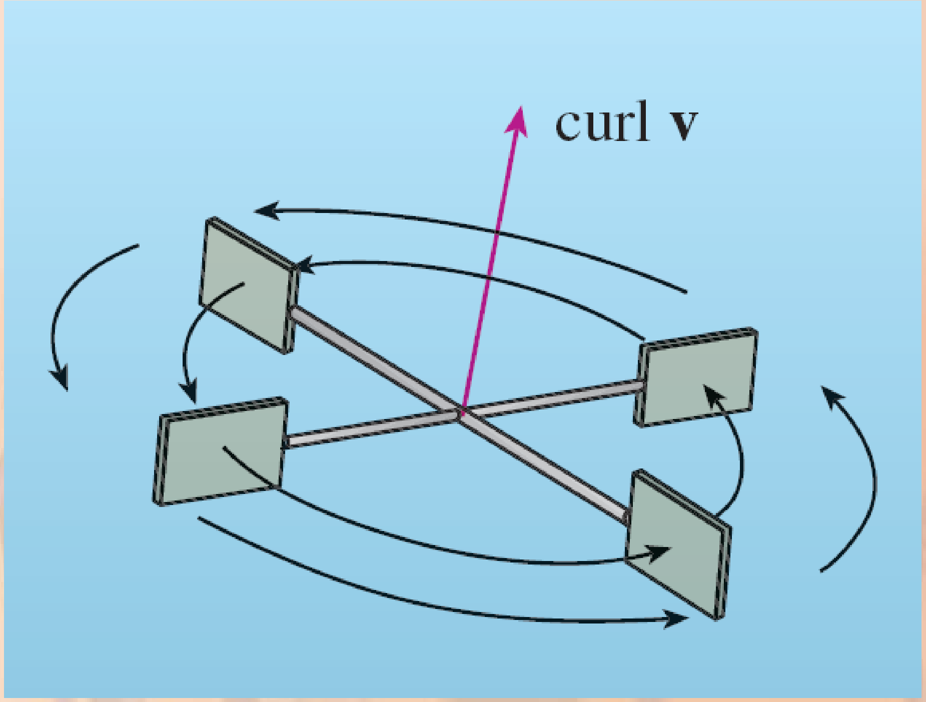
CURL & CIRCULATION

Equation 4 gives the relationship between the curl and the circulation.

- It shows that curl **v** · **n** is a measure of the rotating effect of the fluid about the axis **n**.
- The curling effect is greatest about the axis parallel to curl **v**.

Imagine a tiny paddle wheel placed in the fluid at a point *P*.

- The paddle wheel rotates fastest when its axis is parallel to curl **v**.



16.9

The Divergence Theorem

In this section, we will learn about:

The Divergence Theorem for simple solid regions,

and its applications in electric fields and fluid flow.

INTRODUCTION

In Section 16.5, we rewrote Green's Theorem in a vector version as: $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint\limits_D \operatorname{div} \mathbf{F}(x, y)\, dA$, where *C* is the positively oriented boundary curve of the plane region *D*.

If we were seeking to extend this theorem to vector fields on R^3, we might make the guess that

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint\limits_E \operatorname{div} \mathbf{F}(x, y, z)\, dV$$

where *S* is the boundary surface of the solid region *E*.

It turns out that this is true, under appropriate hypotheses, and is called the Divergence Theorem.

SIMPLE SOLID REGION

We state the Divergence Theorem for regions *E* that are simultaneously of types 1, 2, and 3. We call such regions simple solid regions.

- For instance, regions bounded by ellipsoids or rectangular boxes are simple solid regions.

The boundary of *E* is a closed surface. We use the convention, introduced in Section 16.7, that the positive orientation is outward.

- That is, the unit normal vector **n** is directed outward from *E*.

Let:

- *E* be a simple solid region and let *S* be the boundary surface of *E*, given with positive (outward) orientation.
- **F** be a vector field whose component functions have continuous partial derivatives on an open region that contains *E*.

Then,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

THE DIVERGENCE THEOREM

Thus, the Divergence Theorem states that:

- Under some conditions, the flux of **F** across the boundary surface of *E* is equal to the triple integral of the divergence of **F** over *E*.

The Divergence Theorem is sometimes called Gauss' Theorem after the great German mathematician Karl Friedrich Gauss (1777–1855) (discovered during his investigation of electrostatics).

In Eastern Europe, it is known as Ostrogradsky's Theorem (published in 1826) after the Russian mathematician Mikhail Ostrogradsky (1801–1862).

Find the flux of the vector field

$$\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$$

over the unit sphere

$$x^2 + y^2 + z^2 = 1$$

- First, we compute the divergence of **F**:

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$$

The unit sphere *S* is the boundary of the unit ball *B* given by: $x^2 + y^2 + z^2 \leq 1$

- So, the Divergence Theorem gives the flux as:

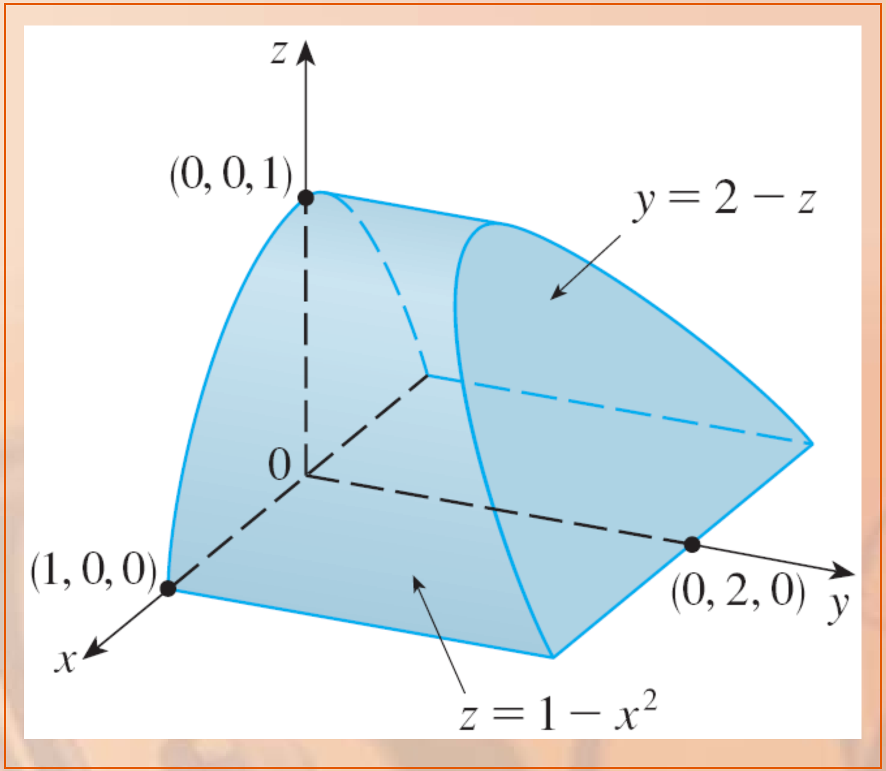
$$\iint_S F \cdot dS = \iiint_B \operatorname{div} \mathbf{F}\, dV = \iiint_B 1\, dV$$

$$= V(B) = \tfrac{4}{3}\pi(1)^3 = \frac{4\pi}{3}$$

Example 2

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where:

- $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + (y^2 + e^{xz^2})\,\mathbf{j} + \sin(xy)\,\mathbf{k}$
- S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, $y + z = 2$



It would be extremely difficult to evaluate the given surface integral directly.

- We would have to evaluate four surface integrals corresponding to the four pieces of *S*.
- Also, the divergence of **F** is much less complicated than **F** itself:

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{\partial}{\partial x}(xy)+\frac{\partial}{\partial y}\left(y^2+e^{xz^2}\right)+\frac{\partial}{\partial z}(\sin xy)\\ &= y+2y=3y\end{aligned}$$

So, we use the Divergence Theorem to transform the given surface integral into a triple integral.

- The easiest way to evaluate the triple integral is to express *E* as a type 3 region:

$$E = \left\{ (x, y, z) \middle| -1 \le x \le 1, 0 \le z \le 1 - x^2, 0 \le y \le 2 - z \right\}$$

Then, we have:

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

$$= \iiint_E \operatorname{div} \mathbf{F} \, dV$$

$$= \iiint_E 3y \, dV$$

$$= 3\int_{-1}^{1} \int_{0}^{1-x^2} \int_{0}^{2-z} y \, dy \, dz \, dx$$

$$= 3\int_{-1}^{1}\int_{0}^{1-x^2} \frac{(2-z)^2}{2}\,dz\,dx$$

$$= \frac{3}{2}\int_{-1}^{1}\left[-\frac{(2-z)^3}{3}\right]_0^{1-x^2} dx$$

$$= -\tfrac{1}{2}\int_{-1}^{1}\left[\left(x^2+1\right)^3 - 8\right]dx$$

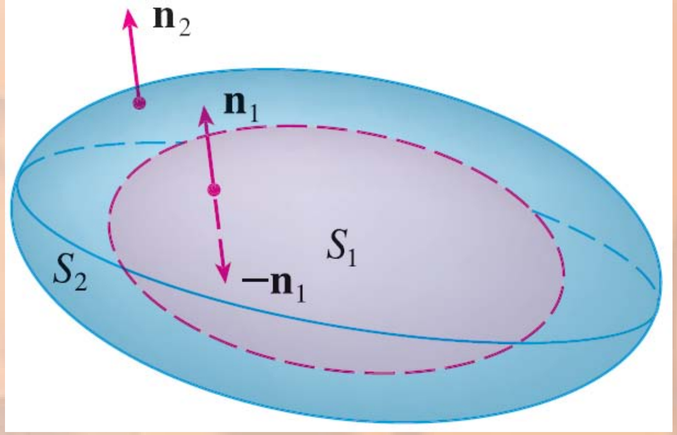
$$= -\int_{0}^{1}\left(x^6 + 3x^4 + 3x^2 - 7\right)dx = \tfrac{184}{35}$$

UNIONS OF SIMPLE SOLID REGIONS

- The Divergence Theorem can also be proved for regions that are finite unions of simple solid regions.
- For example, let's consider the region *E* that lies between the closed surfaces S_1 and S_2, where S_1 lies inside S_2.
  - Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be outward normals of S_1 and S_2.
  - boundary surface of *E* is: $S = S_1 \cup S_2$
  - Its normal $\mathbf{n}$ is given by:

    $\mathbf{n} = -\mathbf{n}_1$ on S_1

    $\mathbf{n} = \mathbf{n}_2$ on S_2



UNIONS OF SIMPLE SOLID RGNS. Equation 7

Applying the Divergence Theorem to *S*, we get:

$$\begin{aligned}\iiint_E \operatorname{div} \mathbf{F}\, dV &= \iint_S \mathbf{F} \cdot d\mathbf{S} \\ &= \iint_S \mathbf{F} \cdot \mathbf{n}\, dS \\ &= \iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1)\, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS \\ &= -\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}\end{aligned}$$

Let's apply this to the electric field (Exp. 5 in Sec.16.1):

$$\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$$

where S_1 is a small sphere with radius *a* and center the origin.

You can verify that div **E** = 0 (Exercise 23).

Thus, the eq. from the previous slide gives:

$$\begin{aligned}\iint_{S_2} \mathbf{E}\cdot d\mathbf{S} &= \iint_{S_1} \mathbf{E}\cdot d\mathbf{S} + \iiint_{E} \operatorname{div}\mathbf{E}\, dV \\ &= \iint_{S_1} \mathbf{E}\cdot d\mathbf{S} \\ &= \iint_{S_2} \mathbf{E}\cdot \mathbf{n}\, dS\end{aligned}$$

The point of this calculation is that we can compute the surface integral over S_1 because S_1 is a sphere.

The normal vector at **x** is **x**/|**x**|.

Therefore.

$$\mathbf{E}\cdot\mathbf{n}=\frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}\cdot\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right)=\frac{\varepsilon Q}{|\mathbf{x}|^4}\mathbf{x}\cdot\mathbf{x}=\frac{\varepsilon Q}{|\mathbf{x}|^2}=\frac{\varepsilon Q}{a^2}$$

since the equation of S_1 is |**x**| = *a*.

Thus, we have:

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n}\, dS = \frac{\varepsilon Q}{a^2} \iint_{S_1} dS$$

$$= \frac{\varepsilon Q}{a^2} A(S_1)$$

$$= \frac{\varepsilon Q}{a^2} 4\pi a^2$$

$$= 4\pi\varepsilon Q$$

APPLICATIONS—ELECTRIC FIELD

This shows that the electric flux of **E** is $4\pi\varepsilon Q$ through any closed surface S_2 that contains the origin.

- This is a special case of Gauss's Law (Equation 11 in Section 16.7) for a single charge.

- The relationship between ε and ε_0 is $\varepsilon = 1/4\pi\varepsilon_0$.

APPLICATIONS—FLUID FLOW

Another application of the Divergence Theorem occurs in fluid flow.

- Let $\mathbf{v}(x, y, z)$ be the velocity field of a fluid with constant density ρ.
- Then, $\mathbf{F} = \rho\mathbf{v}$ is the rate of flow per unit area.

Suppose: $P_0(x_0, y_0, z_0)$ is a point in the fluid.

B_a is a ball with center P_0 and very small radius a.

- Then, div $\mathbf{F}(P) \approx$ div $\mathbf{F}(P_0)$ for all points in B_a since div $\mathbf{F}$ is continuous.

APPLICATIONS—FLUID FLOW

We approximate the flux over the boundary sphere S_a as follows:

$$\begin{aligned}\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} &= \iiint_{B_a} \operatorname{div} \mathbf{F}\, dV \\ &= \iiint_{B_a} \operatorname{div} \mathbf{F}\left(P_0\right) dV \\ &= \operatorname{div} \mathbf{F}\left(P_0\right) V\left(B_a\right)\end{aligned}$$

This approximation becomes better as $a \to 0$ and suggests that:

$$\operatorname{div} \mathbf{F}\left(P_0\right) = \lim_{a\to 0} \frac{1}{V\left(B_a\right)} \iint_{S_a} \mathbf{F} \cdot d\mathbf{S}$$

SOURCE AND SINK

Equation 8 says that div **F**(P_0) is the net rate of outward flux per unit volume at P_0.

(This is the reason for the name divergence.)

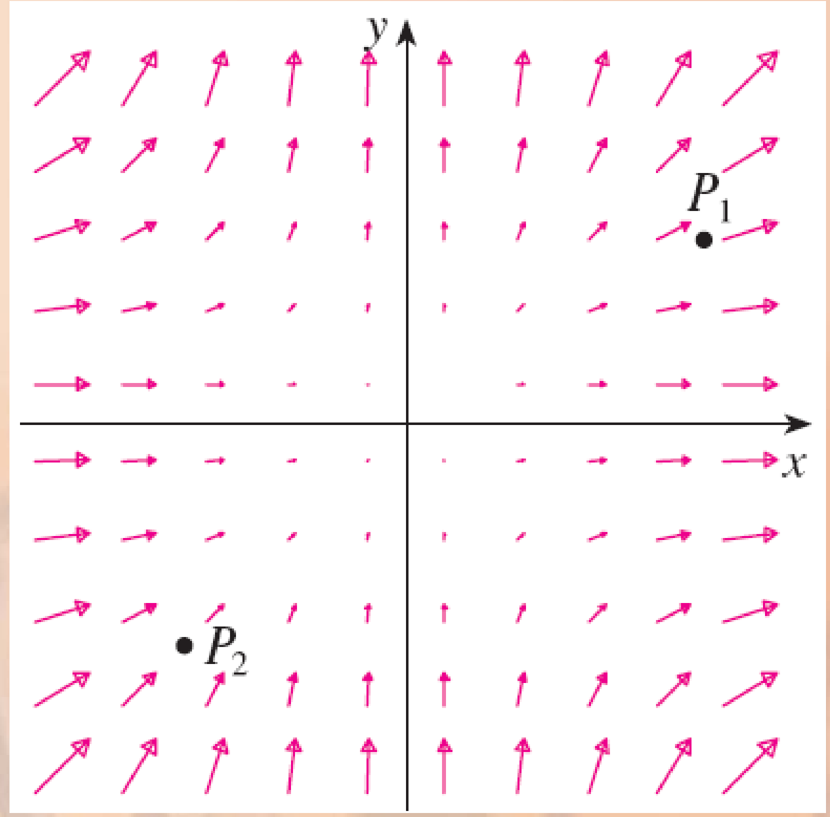
- If div **F**(P) > 0, the net flow is outward near P and P is called a source.

- If div **F**(P) < 0, the net flow is inward near P and P is called a sink.

SOURCE

For this vector field, it appears that the vectors that end near P_1 are shorter than the vectors that start near P_1.

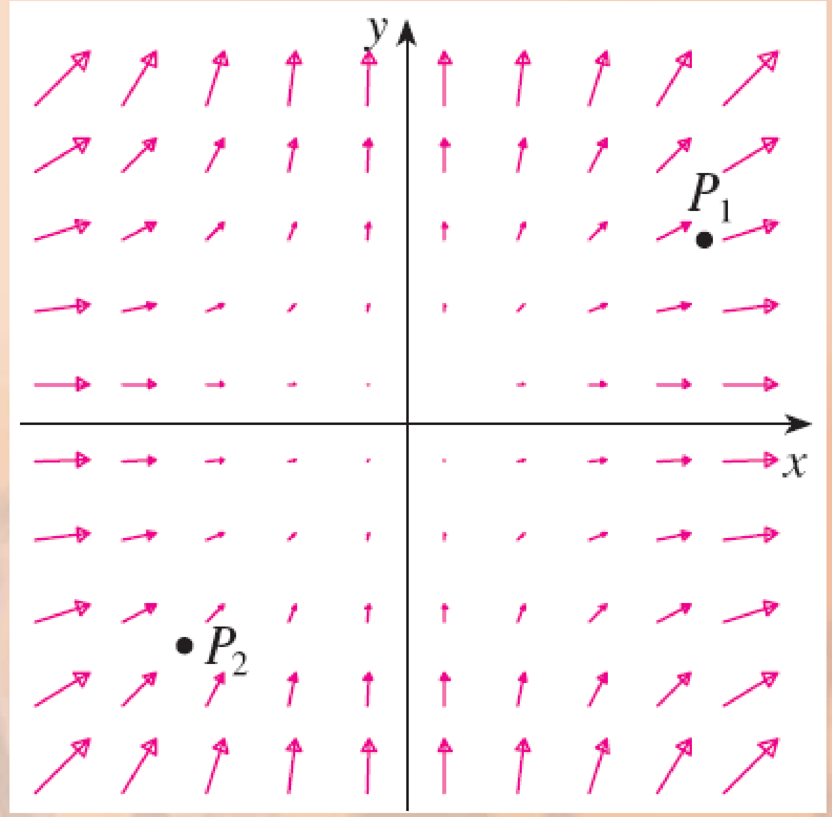
- Thus, the net flow is outward near P_1.
- So, div $\mathbf{F}(P_1) > 0$ and P_1 is a source.



SINK

Near P_2, the incoming arrows are longer than the outgoing arrows.

- The net flow is inward.
- So, div $\mathbf{F}(P_2) < 0$ and P_2 is a sink.



We can use the formula for **F** to confirm this impression.

- Since **F** = x^2 **i** + y^2 **j**, we have div **F** = $2x + 2y$, which is positive when $y > -x$.
- So, the points above the line $y = -x$ are sources and those below are sinks.