HYPERGEOMETRIC FUNCTIONS AND FIBONACCI NUMBERS

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Abstract. This test paper is typeset in the document class **amsart**, without any additional style files. Font size and text size are easy to change; here they are 11pt and 5.5" $\times$ 8.5", respectively. This appears to be quite close to the present format of the *Quarterly*—?.

If we decide to have abstracts for articles in the *Fibonacci Quarterly*, which I think is a good idea, it will look like this in this format.

1. Introduction

This is a shortened and mutilated version of a paper by the first author, for demonstration purposes only. As it stands, it makes no sense; for the real thing, see the *Fibonacci Quarterly* **38** (2000), 342–363.

Hypergeometric functions are an important tool in many branches of pure and applied mathematics, and they encompass most special functions, including the Chebyshev polynomials. There are also well-known connections between Chebyshev polynomials and sequences of numbers and polynomials related to Fibonacci numbers. However, to my knowledge and with one small exception, direct connections between Fibonacci numbers and hypergeometric functions have not been established or exploited before.

It is the purpose of this paper to give a brief exposition of hypergeometric functions, as far as is relevant to the Fibonacci and allied sequences. A variety of representations in terms of finite sums and infinite series involving binomial coefficients are obtained. While many of them are well-known, some identities appear to be new.

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The method of hypergeometric functions works just as well for other sequences, especially the Lucas, Pell, and associated Pell numbers and polynomials, and also for more general second-order linear recursion sequences. However, apart from the final section, we will restrict our attention to Fibonacci numbers as the most prominent example of a second-order recurrence.

2. Hypergeometric Functions

Almost all of the most common special functions in mathematics and mathematical physics are particular cases of the *Gauss hypergeometric series* defined by

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{2.1} $$

where the *rising factorial* $(a)_k$ is defined by $(a)_0 = 1$ and

$$ (a)_k = a(a+1)\cdots(a+k-1), \quad (k \geq 1), \tag{2.2} $$

for arbitrary $a \in \mathbb{C}$. The series (2.1) is not defined when $c = -m$, with $m = 0, 1, 2, \ldots$, unless a or b are equal to $-n$, $n = 0, 1, 2, \ldots$, and $n < m$. It is also easy to see that the series (2.1) reduces to a polynomial of degree n in z when a or b is equal to $-n$, $n = 0, 1, 2, \ldots$. In all other cases the series has radius of convergence 1; this follows from the ratio test and (2.2). The function defined by the series (2.1) is called the Gauss hypergeometric function. When there is no danger of confusion with other types of hypergeometric series, (2.1) is commonly denoted simply by $F(a,b;c;z)$ and called the hypergeometric series, resp. function.

Most properties of the hypergeometric series can be found in the well-known reference works [1], [9] and [8] (in increasing order of completeness). Proofs of many of the more important properties can be found, e.g., in [10]; see also the important works [4] and [11].

At this point we mention only the special case

$$ F(a,b;b;z) = (1-z)^{-a}, \tag{2.3} $$

the binomial formula. The case $a = 1$ yields the geometric series; this gave rise to the term *hypergeometric*.

More properties will be introduced in later sections, as the need arises.

3. Fibonacci Numbers

We will use two different (but related) connections between Fibonacci numbers and hypergeometric functions. The first one is Binet's formula

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \tag{3.1}$$

which allows us to use the identity

$$F\left(a, \frac{1}{2}+a; \frac{3}{2}; z^2\right) = \frac{1}{2z(1-2a)}\left[(1+z)^{1-2a} - (1-z)^{1-2a}\right] \tag{3.2}$$

(see, e.g., [1], (15.1.10)). If we take $a = (1-n)/2$, $z = \sqrt{5}$, and compare (3.2) with (3.1), we obtain

$$F_n = \frac{n}{2^{n-1}} F\left(\frac{1-n}{2}, \frac{2-n}{2}; \frac{3}{2}; 5\right). \tag{3.3}$$

Note that one of the numbers $(1-n)/2$, $(2-n)/2$ is always a negative integer (or zero) for $n \geq 1$, so (3.3) is in fact a finite sum and we need not worry about convergence (see, however, the remark following (4.28)).

Our second approach will be via the well-known connection between Fibonacci numbers and the Chebyshev polynomials of the second kind, namely

$$F_n = (-i)^{n-1} U_{n-1}\left(\frac{i}{2}\right). \tag{3.4}$$

z	$\frac{z}{z-1}$	$1-z$	$1-\frac{1}{z}$	$\frac{1}{z}$	$\frac{1}{1-z}$
5	$\frac{5}{4}$	-4	$\frac{4}{5}$	$\frac{1}{5}$	$-\frac{1}{4}$
$\frac{5}{9}$	$-\frac{5}{4}$	$\frac{4}{9}$	$-\frac{4}{5}$	$\frac{9}{5}$	$\frac{9}{4}$
$\frac{1+\sqrt{5}}{2}$	$\frac{3+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{3-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$
$\frac{2-\sqrt{5}}{4}$	$9-4\sqrt{5}$	$\frac{2+\sqrt{5}}{4}$	$9+4\sqrt{5}$	$-8-4\sqrt{5}$	$-8+4\sqrt{5}$

TABLE 1. Possible arguments

4. Linear and Quadratic Transformations

The next linear transformation formula in the list in [1], p. 559, is

$$\begin{aligned}F(a,b;c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-z)\\ &\quad+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b;c-a-b+1;1-z).\end{aligned} \tag{4.1}$$

However, since $a+b-c=-n$ in (3.3), one of the gamma terms in the numerator is not defined. Instead, we have to use formula (15.3.11) in [1], p. 559, which in the special case where a or b is a negative integer and m is a non-negative integer becomes

$$F(a,b;a+b+m;z) = \frac{\Gamma(m)\Gamma(a+b+m)}{\Gamma(a+m)\Gamma(b+m)}F(a,b;1-m;1-z). \tag{4.2}$$

(For the general case, see, [1], (15.3.11), p. 559). This, applied to (3.3), gives

$$F_n = F\left(\frac{1-n}{2},\frac{2-n}{2};1-n;-4\right). \tag{4.3}$$

Here, we have evaluated the gamma terms in (4.7) as follws, using the duplication formula for $\Gamma(z)$ (see, e.g., [1], p. 256):

$$\begin{aligned}\frac{\Gamma(m)\Gamma(a+b+m)}{\Gamma(a+m)\Gamma(b+m)} &= \frac{\Gamma(n)\Gamma(\frac{3}{2})}{\Gamma(\frac{n}{2}+\frac{1}{2})\Gamma(\frac{n}{2}+1)}\\ &= \frac{(2\pi)^{-1/2}2^{n-1/2}\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}+\frac{1}{2})\frac{1}{2}\sqrt{\pi}}{\Gamma(\frac{n}{2}+\frac{1}{2})\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{2^{n-1}}{n}.\end{aligned}$$

Another transformation formula similar to (4.6) is

$$\begin{aligned}F(a,b;c;z) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F\left(a,1-c+a;1-b+a;\frac{1}{z}\right)\\ &\quad+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}F\left(b,1-c+b;1-a+b;\frac{1}{z}\right).\end{aligned} \tag{4.4}$$

5. An Irreducibility Result

This section is taken from another one of the first author's papers, to illustrate the "theorem" and "proof" environments.

Proposition 5.1. *For any pair of nonnegative integers r and s, the polynomial $B^{(r,s)}_{p-(r+s+1)}(x)$ is irreducible for all primes $p > r+s+1$.*

Proof. For $r = s = 0$, this is a result of Carlitz [6]. Let now $w = r+s+1 > 1$, and let $d_p^{(r,s)}$ denote the least common multiple of the denominators of $B_0^{(r,s)}, \dots, B_{p-w}^{(r,s)}$. By the relation (2.2) we have

$$d_p^{(r,s)} x^{p-w} B_{p-w}^{(r,s)}\left(\frac{1}{x}\right) = \sum_{k=0}^{p-w} \binom{p-w}{k} d_p^{(r,s)} B_k^{(r,s)} x^k, \tag{5.1}$$

and clearly this polynomial has integer coefficients. Now, by Proposition 5.3 we know that $B_k^{(r,s)}$ is p-integral for $0 \le k < p-w$, and that $\alpha_p^{(r,s)}(p-w) = 1$. Hence $p \| d_p^{(r,s)}$. Since p does not divide the binomial coefficients in (6.1), we see that the leading coefficient of the polynomial is not divisible by p, all the other coefficients are divisible by p, but the constant coefficient is not divisible by p^2. Hence Eisenstein's irreducibility ciriterion applies, and the polynomial in (6.1), and consequently $B_{p-w}^{(r,s)}(x)$, are irreducible. □

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