5.6 **LINEAR AND NONLINEAR PERSPECTIVES OF FORECAST ERROR ESTIMATE USING THE FIRST PASSAGE TIME**

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1. INTRODUCTION

Traditionally, the prediction skill of atmospheric models is verified through small amplitude stability analysis. The Lyapunov exponent (LE) and singular vector (SV) decomposition methods are the two popular approaches (e.g. Lorenz, 1984, Dalcher and Kalnay, 1987; Farrell and Ioannou, 1996; Vannitsem and Nicolis, 1997 and others). The model stability is defined as sensitivity to small errors in initial conditions (the first kind of predictability) and measured by an e-folding time computed from the leading LE (or SV).

However, finite amplitude errors exist in many practical cases for example, in medium-range predictions (Vukicevic, 1991; Barkmeijer, 1996) or "imperfect" models (Palmer, 2001 and others). Thus, the analysis should be updated. Herein, one of possible approaches is the probabilistic analysis of forecast error dynamics (e.g., Benzi and Carnavale, 1989, Nicolis 1992; Ehrendorfer 1994 ; Moltineli and Corti, 1998 and others).

Naturally, the knowledge of the probability density function (PDF) of error allows to have the full statistical description of its dynamics. However, herein, even one -dimension error dynamics needs to be studied by numerical methods (Ehrendorfer 1994 ; Nicolis 1992).

The probabilistic approach can be simplified if we only determine the time when the model prediction skill is lost, i.e. forecast error became larger than a given tolerance level (ε). Then, the first passage time can be used as the measure of prediction skill (Ivanov et al., 1994; Ivanov and Margolina, 1999)

In the present paper we take such an approach to illustrate its usefulness for analyzing predictability skill of atmospheric and oceanographic models and further understanding how the finite amplitude error affects low-order characteristics of the model prediction skill.

To do so we use the modified self-consistent model (Nicolis, 1992) for error propagation and the Princeton Oceanographic Model (POM) for shallow water circulation in a semi-closed basin.

2. THE FIRST PASSAGE TIME

We a priori assume that the model dynamics can be described as the stochastic process in an N-dimensional dynamical system and that the characteristics of prediction model skill depending on different physical factors are stochastic. For a stochastic process, the first passage time is defined as the time when the process, starting from a given point (position), reaches a predetermined level for the first time (Gardiner, 1985). It was suggested by Ivanov et al, 1994, Ivanov and Margolina, 1999 to use the first passage time as one of measures of the model prediction skill.

There are several reasons to do so. First, the first passage time characterizes both linear and nonlinear perspectives of forecast error. In the case of small forecast error and for non stochastically forcing dynamics the mean passage time coincides with the e-folding time. Second, the statistical moments of the first passage time can be easily calculated by an iterative technique developed in Ivanov et al., 1999. Third, in many practical applications such an approach allows to get analytical estimations of the model prediction skill. Fourth, this approach is universal for studying both the first and second kinds of predictability.

Because the first passage time is stochastic we can introduce a special function characterizing the model prediction skill. This is the probability density of prediction (PDP) $P(t_0, \xi^0, t-t_0)$ which is the probability that the value of forecasting error ξ satisfies the inequality

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$\left\langle \left\| \xi^t \mathbf{A} \xi \right\| \right\rangle \leq \varepsilon^2$ for the time period $(t - t_0)$ dt when it equals ξ_0 at $t = t_0$. Herein, **A** is the scaling matrix, the bracket denotes the ensemble average over realizations generated by stochastic forcing and/or uncertain initial condition. The surface S determined by $\left\langle \left\| \xi^t \mathbf{A} \xi \right\| \right\rangle = \varepsilon^2$ is the ellipsoid limiting the domain of possible variations of forecast error.

The PDP doesn't coincide with usual probability density function and satisfies the Pontryagin - Kolmogorov equation (PKE) (Pontryagin et al., 1968, Ivanov et al., 1994):

$$LP(t_0, \xi^0, t - t_0) = 0 \qquad (2.1)$$

where $L = \frac{\partial}{\partial t} - u_n \nabla_n - \chi_{nm} \nabla_n \nabla_m, \nabla_n = \frac{\partial}{\partial \xi_n^0}$.

u_n, χ_{nm} are determined from an evolution law and stochastic forcing of hydrodynamic model (Ivanov et al., 1994), n,m=1,…,N.

PDP should satisfy the following initial and boundary conditions:

$$P(t_0, \xi^0, 0) = 1,$$

$$P(t_0, \xi^0, t - t_0)\Big|_{\xi^0 \subset S} = 0,$$

(losing the model skill when forecast error reaches S),

or $$\chi_{nm} \frac{\partial P}{\partial \xi_m^0} = 0$$

(impossibility of error to be less than some level, determined by thermal or another kinds of noise distorted the model dynamics).

Note both boundary conditions can be simultaneously applied on different parts of S.

The probability density of prediction can be effectively calculated by numerical methods, for example, through the ensemble prediction technique (EPT) or by the special iteration approach developed in Ivanov e t al., 1999.

The PDP plays a key role for the estimation of the model prediction skill. Its knowledge allows to determine the moment of first passage time as

$$\tau_k(\xi^0) = k \int_{t_0}^{\infty} P(t_0, \xi^0, t - t_0) \left(\mathrm{t} - \mathrm{t}_0\right)^{\mathrm{k}-1} d\,\mathrm{t} \qquad (2.2)$$

A low-order characteristics of the prediction skill are the mean and variance of predictability time $\langle \tau \rangle = \tau_1$ and $\langle \delta\tau^2 \rangle = \tau_2 - \tau_1^2$ respectively.

Here, the brackets denote the ensemble average over realizations generated by stochastic forcing.

The prediction skill for atmosphere model is currently verified using the perfect model concept with uncertainty appearing only in the initial conditions. In our approach, the uncertainty is easily taken into account through additional averaging of (2.2) with respect to an ensemble of initial error.

3. SELF-CONSISTENT ERROR FORECAST MODEL

Nicolis (1992) estimated the prediction skill of Lorenz (1985) three-component atmospheric model through the self-consistent error prediction model. This model was obtained through projection of full error model onto the unstable manifold and replacement of the reference circulation to special stochastic term. The model is written by

$$\frac{d\xi}{dt} = (\sigma - g\xi)\xi + \nu(t)\xi, \quad \xi\Big|_{\mathrm{t}=\mathrm{t}_0} = \xi^0, \qquad (3.1)$$

where $\xi \subset [0, \infty)$ is non-dimensional amplitude of error, σ and *g*, are generally time-independent parameters whose properties depend on the underlying model attractor, $\langle \nu(t) \rangle = 0, \ \langle \nu(t)\nu(t') \rangle = q^2 \delta(t - t')$.

We modified the self-consistent model because in the real world any dynamics is distorted by at least the thermal (molecular) noise and value of initial error ξ^0 cannot be less than the amplitude of the thermal noise ξ_{th}.

Let us account this noise and introduce the special scale $\xi_{th} = \left(\langle I_{therm} \rangle\right)^{1/2}$, I_{therm} is the intensity of the thermal noise. The contribution of ξ_{th} to the predictability time can very small if $\xi^0 >> \xi_{th}$. However, use of ξ_{th} leads to a correct asymptotic of the predictability time for $\xi^0 \to 0$. The introduced scale also can be interpreted as the characteristic scale of unresolved sub-grid motions.

Although the model (3.1) is simple it demonstrates several dynamical regimes of error behavior. The non-dimensional parameter $\lambda = 2\sigma / q^2$ is a threshold between different dynamical regimes of error evolution.

For $\lambda > 1$, $\xi_{th} << 1$ and $g << 1$ we obtained the e-folding time written as

$$\langle \tau \rangle \approx \frac{2}{q^2(\lambda - 1)} \ln\left(\frac{\varepsilon}{\xi_0}\right) \qquad (3.2)$$

for the critical point $\lambda = 1$ the dependence for the predictability time differs from the e -folding time

$$\langle \tau \rangle = \frac{1}{q^2} \ln(\frac{e}{\xi^0}) \left[\ln(\varepsilon \xi^0) - 2 \ln(\xi_{th}) \right] \quad (3.3)$$

which demonstrates that reduction of any noise intensity increases the predictability time but only on the order of $\sim \ln(\delta_{th})$.

Lower the critical point $(\lambda > 1)$ the predictability time can be determined as

$$\langle \tau \rangle = \frac{1}{(1-\lambda) q^2} \left[\beta B(\frac{\varepsilon}{\xi^0}) - \ln(\frac{\varepsilon}{\xi^0}) \right] \qquad (3.4)$$

where $B(\frac{\varepsilon}{\xi^0}) = \frac{\varepsilon^{1-\lambda} - \left(\xi^0\right)^{1-\lambda}}{\varepsilon^{1-\lambda} \left(\xi^0\right)^{1-\lambda}}$, $\beta = \frac{(\lambda+1)}{\xi_{th}^{1-\lambda}}$.

The effect of nonlinear terms in (3.1) appears in the power terms in (3.2)-(3.4), which reduce the predictability time and increasing it variance. Detailed results are presented in Chu et al., 2001a.

4. PREDICTION SKILL OF POM

For chosen mean wind stress $\tau_w = 10^{-3} m^2 s^{-2}$ and basin geometry (a rectangular basin with the geometrical sizes of $1000km \times 1050km \times 2km$, plane bottom and a single open boundary) we found that for 50-60 days the flow starting from the state of rest develops in the quasi-stationary inferred circulation. We ignored the horizontal diffusion, and only the bottom friction was considered. The details of numerical experiments can be found in Chu et al., 2001 b.

The non-stationary circulation developed within a two-month period was chosen as the reference solution for the study of the POM stability. We focused only on the short and intermediate predictions limited by 2-2.5 months. The reference solution were distorted by stochastic errors of initial conditions, stochastic wind and normal velocity along the open boundary.

We calculated numerically by EPT and analytically from PKE the probability density of prediction, the predictability time and its variance.

The mean and the most probable times, the variance of the mean time are the local characteristics of the probabilistic model stability in the perturbation geostrophic stream function L_2 norm .

In order to determine the prediction skill, the POM was reduced to a simpler stochastic dynamical system. The system is a set of stochastic ordinary differential equations whose parameters are embedded in probabilistic properties of the full model.

We have developed the set of orthogonal basis functions (normal modes) for semi-closed seas. That allowed constructing a dynamical system for the full POM.

To do so we represent velocity vector through the two scalar potentials Ψ and Φ (Chu et al., 2001 b) as

$$\mathbf{u} = \nabla \times (\mathbf{k} \Psi) + \nabla \Phi \qquad (4.1)$$

and chosen the following open boundary condition for the velocity potential

$$\Phi = 0 \qquad (4.2)$$

Such an approach allows to identify Ψ as the geostrophic steam functions and exclude the velocity potential from the analysis because ratio of mean kinetic energy correspond to potential velocity and mean kinetic energy of irrotational motions was less than 0.01.

Then, we introduced a new variables by

$$\Psi' = \Psi - \int_0^l u_{norm} d\sigma \qquad (4.3)$$

where u_{norm} is the normal velocity along the open boundary.

The transformation (4.3) allowed to reformulate the POM stability problem into uncertainty of normal velocity along an open boundary as the probabilistic stability of a dynamical system with multiplicative stochastic forcing and to introduce the spectral representations

$$\Psi' = \sum_{m=1}^{M} A_m \Psi_m \qquad (4.4)$$

where Ψ_m is the eigenfunctions of Laplacian operator.

We have found that the 10- dimensional dynamical system for A_m approximated the reference solution with the accuracy around 1-2 %. In comparing with ten normal modes three empirical orthogonal functions contents more than 0.1% of total kinetic energy of the reference circulation.

Then, PDP was calculated for the full model, for dynamical systems of 10^{th}(M=10)and 100^{th} (M=100) orders by EPT and only for a 10-

dimensional dynamical system from PKE. We vary ensemble size between 50 and 10000 members.

We have found that the forecast error can be strongly reduced on the initial growing stage. The zero growth (or even negative) of forecast error within 30-day period is obtained if Gaussian stochastic noise with the correlation radius equaled to 300 km was added into initial conditions and uncertainty of open boundary conditions was small. Reducing the correlation radius, i.e. transiting to white noise, we reanimated effective growth of initial perturbations.

The prediction skill of the POM is the most sensitive to the systematic errors of normal velocity along the open boundary. Stochastic perturbations of the normal velocity can often result only in super-slow growth of the forecasting error.

Another interest physical result is that the spectrum of forecast error collapse fast into low-frequency domain of wave numbers. The physical reason of this phenomena is geostrophic adjuction of the reference circulation.

Three typical scenarios of the predictability process have been found.

Scenario 1. After a rapid growth of initial perturbations a non Gaussian probability density function (PDF) for the first passage time had a symmetric form with narrow variance (Figure 1 a,d,e; $\varepsilon = 10^{-1}$). Such a scenario is the most probable for large forecast error. Herein, the ratio of the mean predictability time to the most probable (α) is around 1. The mean predictability time calculated by EPT, Singular vectors and from PKE was equal to 27.6, 26.9 and 25.6 days, respectively. The variance of this time written in the same notations isl 11.7, - and 10.5 day^2. The prediction skill of the model could be considerably improved through the filtration of the fastest growing initial perturbations .

Scenario 2. We illustrate it through the analysis of flow forced by stochastic wind with the variance equaled to 21 m^2/s^2 (Fig.2 b,e,h ; $\varepsilon = 10^{-3}$). Herein, the initial condition was weakly perturbed.

PDF had a non- symmetric form with a long tail in domain of the long-term prediction, $\alpha \sim 1.3$. The calculated mean and variance of the first-passage time, $\langle\tau\rangle(\langle\delta\tau^2\rangle)$, are 1.32 (0,31); and 1.31 (0.33), respectively. Although the mean predictability time is not large, the predictability time for single realizations can be very large.

We called it "extreme prediction", which strongly depends on properties of the model attractor and weakly on uncertainty of initial conditions.

Notice that such a scenario is often take place for small values of tolerance ε, and indicates a linear evolution of forecast error. Improvement of the model prediction skill requires some different approaches from reduction of initial errors.

Scenario 3. This scenario is for the case that only uncertainty of open boundary condition exists. Instead of uniqueness, PDP had several maximuma, $\alpha \approx 5.2$ (Fig. 1 c, f, i; $\varepsilon = 5.10^{-3}$). Use of the notations in scenario 2, $\langle\tau\rangle(\langle\delta\tau^2\rangle)$ was equal to 10.6 (84.3) and 12.1 (88.3), respectively. Herein, we have both short-term and long-term (extreme) predictions.

Notice that the uncertainty of initial conditions plays a second role again. The tail of PDF can be directed along both short and long-term predictions.

The principle property obtained from the scenario is abnormal large value of the variance of the predictability time. Numerical calculations demonstrate that the scenario 3 can be realized for both the small and large values of forecast errors.

5. THE EXTREME PREDICTIONS IN OCEAN

In order to check existence of extreme predictions found numerically in the real world we estimated the prediction skill one of high-resolution circulation models of the Gulf of Mexico. This model was developed by Kantha and Clayson, 2000.

The model was integrated on 15X15 km grid and forced by monthly wind and assimilated climatic boundary conditions in Yakatan Channel and Strait of Florida. The model prediction skill was estimated using difference between trajectories of buoys and of synthetic particles deployed in numerically simulated circulation.

We introduced the mean predictability time averaged over buoy trajectories and by all buoys. Number of terms of such an ensemble achieved 1600.

Our calculations clearly show the extreme predictions. For example, for 50- km tolerance level, and the most probable time equals 3 days, the extreme prediction time can reach 12-15 days (Fig.2 a, b).

The mean predictability time linearity grows with increasing the value of tolerance. For values of tolerance are more than 50 km we had 30-50 day successful predictions of buoy trajectories.

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