BACKWARD FOKKER-PLANCK EQUATION FOR DETERMINATION OF MODEL PREDICTABILITY WITH UNCERTAIN INITIAL ERRORS

Peter C. Chu, and Leonid M. Ivanov
Department of Oceanography
Naval Postgraduate School
Monterey, California

1. INTRODUCTION

It is widely recognized that uncertainty in atmospheric and oceanic models can be traced back to two factors (Lorenz 1984a, 1987). First, in defining the state of atmosphere (or ocean), a number of errors are involved arising from the finite resolution of measurement or from discretization in a numerical experiment, as a result of which small-scale “subgrid” processes are either discarded or parameterized. Second, once present, small errors of the kind mentioned above trigger a complex response leading to their subsequent amplification. The model predictability versus boundary condition error was discussed by Chu (1999) using the Lorenz system.

The model predictability can be measured by two parameters: instantaneous error (IE) and predictability time (PT). The IE and PT are used for models with and without given initial condition errors, respectively.

The IE measure is widely used for model evaluation. The predictability is regarded as the model error growth due to the initial condition error. This implies that the initial condition error should be given. The evaluation process becomes to study the stability of the dynamical system with a given initial condition error and to determine either the leading (largest) Lyapunov exponent (e.g., Lorenz 1969) or the amplification factors calculated from the leading singular vectors (e.g., Farrell and Ioannou 1996 a, b). It is well known that the stability analysis using the Lyapunov exponents and the singular vectors is not unique (Has'minskii, 1980). Probabilistic stability analysis becomes available in practical application (Ehrendorfer 1994 a, b; Nicolis 1992). The statistical properties of the prediction error are described through the probability density function (PDF) satisfying the Liouville equation or the Fokker-Plank equation. Solving this equation, we may calculate the mean and variance of errors. Nicolis (1992) explored the probabilistic properties of error-growth dynamics in the atmosphere using a simple low-order model (displaying a single positive Lyapunov exponent) giving rise to chaotic behavior. A large number of numerical experiments were performed to assess the relative importance of average and random elements in error growth.

In this study, we first develop a theoretical framework for predictability evaluation using the PT measure, and then to illustrate its usefulness using the one-dimensional probabilistic error growth model proposed by Nicolis (1992).

2. PREDICTABILITY ERROR

2.1. Dynamical System with Stochastic Forcing

Let $\mathbf{x}(t) = [x^{(1)}(t), x^{(2)}(t), \ldots, x^{(n)}(t)]$ be the full set of variables characterizing the dynamics of the atmosphere (or ocean) in a certain level of description. We embed the evolution of $\mathbf{x}(t)$ in a phase space spanned by the whole set of variables $[x^{(1)}(t), x^{(2)}(t), \ldots, x^{(n)}(t)]$. Let deterministic law be given by

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t)$$

where **f** is a functional. With a stochastic forcing **q**, Eq.(1) becomes a stochastic differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) + q(t)\mathbf{x} \qquad (1)$$

Atmospheric (or ocean) prediction is to find the solution of (2) with an initial condition

$$\mathbf{x}(t_0) = \mathbf{x}_0 \qquad (2)$$

where $\mathbf{x}_0$ is an initial particle position in the phase space. For simplicity, we assume in this paper that the stochastic forcing **g** is a white noise with zero mean

$$\langle q(t) \rangle = 0 \qquad (3)$$

and variance q^2

$$\langle q(t)\, q(t') \rangle = q^2 \delta(t - t') \quad (4)$$

where the bracket < > is defined as ensemble mean over realizations generated by the stochastic forcing $q(t)$, and δ is the Delta function.

2.2. Prediction Model and Error Variance

Let $\mathbf{y}(t) = [y^{(1)}(t), y^{(2)}(t), \ldots, y^{(n)}(t)]$ be the prediction of $\mathbf{x}(t)$ using a prediction model

$$\frac{d\mathbf{y}}{dt} = \mathbf{h}(\mathbf{y}, t) \qquad (5)$$

with an initial condition

$$\mathbf{y}(t_0) = \mathbf{y}_0 . \qquad (6)$$

Difference between reality ($\mathbf{x}$) and prediction ($\mathbf{y}$) at any time t ($>t_0$)

$$\mathbf{z} = \mathbf{x} - \mathbf{y}$$

is defined as the prediction error vector and this difference at t_0

$$\mathbf{z}_0 = \mathbf{x}_0 - \mathbf{y}_0$$

is defined as the initial error vector. If the components $[x^{(1)}(t), x^{(2)}(t), \ldots, x^{(n)}(t)]$ are not equally important in terms of prediction, the uncertainty of model prediction can be measured by the error variance

$$\mathrm{Var}(\mathbf{z}) = \langle \mathbf{z}'\mathbf{A}\mathbf{z} \rangle \qquad (7)$$

with an $n \times n$ diagonal weight matrix

$$\mathbf{A} = \begin{bmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & A_n \end{bmatrix} . \qquad (8)$$

where $A_1, A_2, \ldots A_n$ are non-negative real numbers, which represent relative importance of the corresponding components. Here, the superscript prime indicates the transpose of the prediction error vector $\mathbf{z}$.

3. TWO TYPES OF MODEL PREDICTABILITY MEASURES

3.1. IE Measure

Use of the IE measure is usually associated with the investigation of model error growth with a given initial error ζ_0

$$|\mathbf{z}_0| = \zeta_0 \qquad (9)$$

For example, Nicolis (1992) defined the ensemble mean IE

$$u_\zeta(t) = \frac{1}{2\mu(\Gamma)} \int_\Gamma d\mathbf{x}_0 \int_\Gamma d\mathbf{y}_0 \, |\mathbf{z}(t)| \, \delta(|\mathbf{z}(t_0)| - \zeta_0) \qquad (10)$$

to evaluate the predictability of the Lorenz attractor (1984). Here, Γ denotes the phase-space region belonging to the attractor; $\zeta = |\mathbf{z}|$; the factor (½) accounts for the symmetry between $\mathbf{x}$ and $\mathbf{y}$; and $\mu(\Gamma)$ is the volume of the attractor. Since the dynamical system (1) contains stochastic forcing, the ensemble mean IE is a random variable with a probability function $\hat{P}(\zeta, t)$, which satisfies the Fokker-Planck equation (Nicolis 1992)

$$\frac{\partial \hat{P}}{\partial t} + \mathbf{f}(\mathbf{x}, t)\frac{\partial \hat{P}}{\partial \mathbf{x}} = \frac{q^2}{2}\frac{\partial^2}{\partial \mathbf{x}\partial \mathbf{x}}(\mathbf{x}^2 \hat{P}) \qquad (11)$$

3.2. PT Measure

Without knowing the initial error ζ_0, we cannot use the IE measure to evaluate the model predictability. At that case, we may first define a scalar parameter, ε, called the tolerance prediction error. The prediction is meaningful only if the error variance satisfies the following inequality

$$\mathrm{Var}(\mathbf{z}) \le \varepsilon^2 \qquad (12)$$

which represents an ellipsoid $S_\varepsilon(t)$ in the n-dimensional phase space (Fig. 1), which is called the predictability ellipsoid. At any instance t, a valid model prediction is represented by a time period $(t - t_0)$ that the stochastic trajectory $\mathbf{z}$ is still in the ellipsoid $S_\varepsilon(t)$. We may define $(t - t_0)$ as the valid prediction period. Obviously, this parameter is a random variable since the trajectory $\mathbf{z}$ is stochastic. The joint probability density function of $(t - t_0)$ and $\mathbf{z}_0$ satisfies the backward Fokker-Planck equation (Section 3.6 in Gardiner 1983)

$$\frac{\partial P}{\partial t} - [\mathbf{f}(\mathbf{z}_0, t)]\frac{\partial P}{\partial \mathbf{z}_0} - \frac{1}{2}q^2 \mathbf{z}_0^{\,2}\frac{\partial^2 P}{\partial \mathbf{z}_0 \partial \mathbf{z}_0} = 0 \qquad (13)$$

Suppose the prediction error vector $\mathbf{z}$ moving in the phase space (i.e., the prediction errors are not constant), and suppose the tolerance prediction error ε to be sufficiently small. The probability for $\mathbf{z}_0$ in the ellipsoid $S_\varepsilon(t)$ is zero (Fig. 1),

$$P(t_0, \mathbf{z}_0 \in S_\varepsilon(t), t - t_0) = 0 \qquad (14)$$

which is the boundary condition of (13). The probability for the prediction error vector at $t = \infty$ is zero,

$$\lim_{t\to\infty} P(t_0, \mathbf{z}_0, t-t_0) = 0$$

and the temporal integration of $P(t_0, \mathbf{z}_0, t - t_0)$ from t_0 to ∞ should be one,

$$\int_{t_0}^{\infty} P(t_0, \mathbf{z}_0, t-t_0)dt = 1 \qquad (15)$$

Since $P(t_0, \mathbf{z}_0, t - t_0)$ is the probability of the valid prediction period $(t - t_0)$ with the initial error vector $\mathbf{z}_0$ at t_0, its first moment

$$\tau_1(\mathbf{z}_0) = \int_{t_0}^{\infty} P(t_0, \mathbf{z}_0, t-t_0)(t-t_0)dt$$

(16) denotes the ensemble mean of the valid prediction period, or called PT. Its second moment

$$\tau_2(\mathbf{z}_0) = \int_{t_0}^{\infty} P(t_0, \mathbf{z}_0, t-t_0)(t-t_0)^2 dt$$

(17) indicates the variation of the valid prediction period, or called VPT. The uncertainty of initial condition errors is easily taken into account through additional averaging of (16) and (17) over an ensemble of initial perturbations.

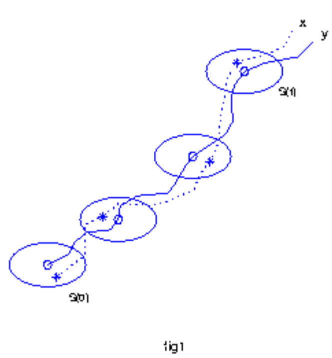


Figure 1. Phase space trajectories of model prediction y (solid curve) and reality x (dashed curve). The positions of reality and prediction trajectories at time instance are denoted by "*" and "o ", respectively.

3.3. PT and VPT Equations for an Autonomous System

Usually, two steps are used for computing PT and VPT: (1) to obtain $P(t_0, \mathbf{z}_0, t - t_0)$ through solving the backward Fokker-Planck equation (13), and (2) to compute PT and VPT using (16) and (17). For an autonomous dynamical system,

$$\mathbf{f} = \mathbf{f}(\mathbf{z}_0)$$

the procedure is much simpler. We multiply the backward Fokker-Planck equation (13) by $(t - t_0)$ and $(t - t_0)^2$ respectively, integrate both equations with respect to t from t_0 to ∞, and obtain the PT equation

$$\mathbf{f}(\mathbf{z}_0)\frac{\partial \tau_1}{\partial \mathbf{z}_0} + \frac{q^2\mathbf{z}_0^2}{2}\frac{\partial^2 \tau_1}{\partial \mathbf{z}_0 \partial \mathbf{z}_0} = -1 \qquad (18)$$

and the VPT equation

$$\mathbf{f}(\mathbf{z}_0)\frac{\partial \tau_2}{\partial \mathbf{z}_0} + \frac{q^2\mathbf{z}_0^2}{2}\frac{\partial^2 \tau_2}{\partial \mathbf{z}_0 \partial \mathbf{z}_0} = -2\tau_1 . \qquad (19)$$

Both (18) and (19) are linear. Usually, they have analytical solutions.

4. A SELF-CONSISTENT PROBABILISTIC ERROR GROWTH MODEL

To illustrate similarity and dissimilarity between PT and IE measures, we use the one-dimensional probabilistic error growth model (Nicolis 1992)

$$\frac{d\xi}{dt} = (\sigma - g\xi^2) + v(t)\xi , 0 \le \xi < \infty \qquad (20)$$

which is the projection of a self-consistent stochastic error growth model [from the Lorenz attractor (Lorenz 1984)] into the unstable eigenvector ξ corresponding to the positive Lyapunov exponent σ.

$$\langle v(t)\rangle = 0, \ \langle v(t)v(t')\rangle = q^2\delta(t-t') . \qquad (21)$$

The parameters in the error growth model (20) are chosen by Nicolis (1992) as

$$\sigma = 0.64, \ g = 0.3 , \ q^2 = 0.2 . \qquad (22)$$

5. EVALUATION OF MODEL PREDICTABILITY WITH UNKNOWN INITIAL CONDITION ERROR

5.1. PT and VPT Equations

Due to limited sampling, the initial condition errors and its PDF are usually uncertain. Without knowing $\hat{P}(\xi,0)$ and $\langle\xi\rangle|_{t=0}$, the IE measure is hardly used in predictability evaluation. However, the PT measure is easily used.

For the probabilistic error growth model (20), the PDF for the random variable $(t -$

t_0), [i.e., $P(t_0, \xi_0, t - t_0)$] satisfies the backward Fokker-Planck equation,

$$\frac{\partial P}{\partial t} - [\sigma - g\xi_0^{\,2}]\frac{\partial P}{\partial \xi_0} - \frac{1}{2}q^2 \frac{\partial^2 P}{\partial \xi_0^{\,2}} = 0 \qquad (27)$$

and the ellipsoid (12) becomes an interval,

$$0 \le \xi_0 \le \varepsilon .$$

The PT and VPT equations (18) and (19) become ordinary differential equations

$$(\sigma\xi_0 - g\xi_0^2)\frac{d\tau_1}{d\xi_0} + \frac{q^2\xi_0^2}{2}\frac{d^2\tau_1}{d\xi_0^{\,2}} = -1 \qquad (28)$$

$$(\sigma\xi_0 - g\xi_0^2)\frac{d\tau_2}{d\xi_0} + \frac{q^2\xi_0^2}{2}\frac{d^2\tau_2}{d\xi_0^{\,2}} = -2\tau_1 \qquad (29)$$

When the initial condition error reaches the tolerance level, $\xi_0 = \varepsilon$, the model capability of prediction is lost no matter how good the model is, and τ_1 and τ_2 become zero,

$$\tau_1 = 0,\ \tau_2 = 0 \ \text{ for } \xi_0=\varepsilon. \qquad (30)$$

When the initial condition error is below the noise level, ξ_{noise}, the model capability of prediction does not depend on the initial condition, and τ_1 and τ_2 do not depend on ξ_0,

$$\frac{d\tau_1}{d\xi_0} = 0, \qquad \frac{d\tau_2}{d\xi_0} = 0 \qquad \text{for} \qquad \xi_0=\xi_{\text{noise}} \qquad (31)$$

5.2. Dependence of PT and VPT on $(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon)$

Solving the second-order differential equations (28) and (29) with the boundary conditions (30) and (31), we obtain an analytic solutions for τ_1

$$\tau_1(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon) = \frac{2}{q^2}\int_{\bar{\xi}_0}^{1} y^{-\frac{2\sigma}{q^2}} \exp(\frac{2\varepsilon g}{q^2}y) \left[\int_{\bar{\xi}_{noise}}^{y} x^{\frac{2\sigma}{q^2}-2} \exp(-\frac{2\varepsilon g}{q^2}x)dx\right]dy \qquad (32)$$

and for τ_2

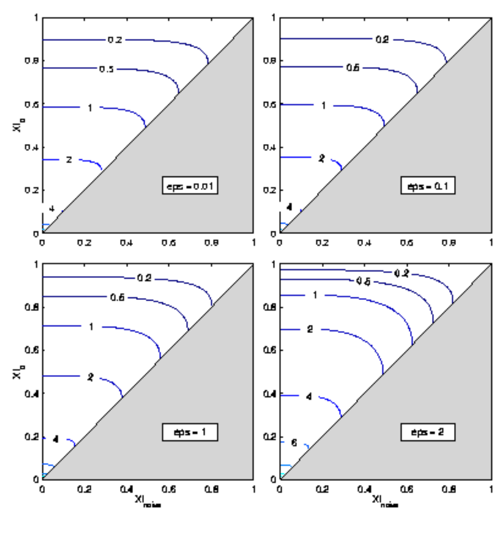
$$\tau_2(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon) = \frac{4}{q^2}\int_{\bar{\xi}_0}^{1} y^{-\frac{2\sigma}{q^2}} \exp(\frac{2\varepsilon g}{q^2}y) \left[\int_{\bar{\xi}_{noise}}^{y} \tau_1(x) x^{\frac{2\sigma}{q^2}-2} \exp(-\frac{2\varepsilon g}{q^2}x)dx\right]dy \qquad (33)$$

where

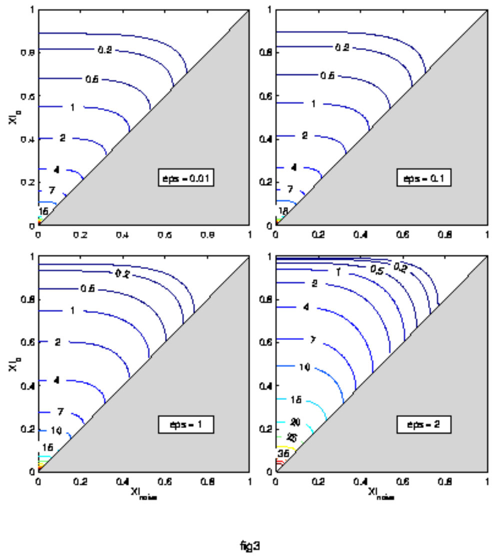
$$\bar{\xi}_0 = \xi_0/\varepsilon, \qquad \bar{\xi}_{noise} = \xi_{noise}/\varepsilon$$

are the initial condition error and noise level scaled by the tolerance error ε.

Figures 2 and 3 show the contour plots of $\tau_1(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon)$ and $\tau_2(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon)$ versus $(\bar{\xi}_0, \bar{\xi}_{noise})$ for four different values of ε (0.01, 0.1, 1, and 2). Both τ_1 and τ_2 are almost independent on the random noise $\bar{\xi}_{noise}$ (contours are almost paralleling to the horizontal axis) when the initial error ($\bar{\xi}_0$) is larger, and are almost independent on the tolerance error ε when ε ≤ 0.1.



Figures 2. Contour plots of $\tau_1(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon)$ versus $(\bar{\xi}_0, \bar{\xi}_{noise})$ for four different values of ε (0.01, 0.1, 1, and 2) using Nicolis model with stochastic forcing.



Figures 3. Contour plots of $\tau_2(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon)$ versus $(\bar{\xi}_0, \bar{\xi}_{noise})$ for four different values of ε (0.01, 0.1, 1, and 2) using Nicolis model with stochastic forcing.

6. CONCLUSIONS

(1) There are two model predictability measures: instantaneous error and predictability time. The instantaneous error measure is used for models with a given initial condition error. The predictability time measure is used for models with uncertain initial condition error.

(2) We developed a theoretical framework to determine the predictability time (τ_1) and its variability (τ_2) for nonlinear stochastic dynamical system. The initial condition error ($\mathbf{z}_0$) and the valid prediction period ($t - t_0$) are treated as random variables. The joint probability density function, $P(t_0, \mathbf{z}_0, t - t_0)$, satisfies the backward Fokker-Planck equation when the random variable ($t - t_0$) is assumed homogeneous. This equation has two independent variables (t, $\mathbf{z}_0$) and given initial and boundary conditions (well-posed). After solving the backward Fokker-Planck equation, it is easy to obtain τ_1 and τ_2 since they are the ensemble mean and variance of ($t - t_0$), respectively.

(3) For an autonomous dynamical system, we derived time-independent second-order linear differential equations for τ_1 and τ_2 with given boundary conditions. This is a well-posed problem and the solutions are easily to be obtained.

(4) For the one-dimensional probabilistic error growth model (Nicolis 1992), the second-order ordinary differential equations of τ_1 and τ_2 have analytical solutions, which show the following features of τ_1 and τ_2: (a) decreasing with increasing initial condition error (or with increasing random noise), (b) increasing with the increasing tolerance error ε, (c) almost independent on ε with small tolerance error ($\varepsilon \leq 0.1$).

(5) The stochastic forcing, acting as a multiplicative white noise, reduces the Lyapunov exponent (increase predictability time) and stabilizes dynamical system (i.e., Nicolis' one-dimensional probabilistic error growth model). On the other hand, the random noise decreases the predictability time and destabilizes the dynamical system.

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