How Long Can an Atmospheric Model Predict?

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Abstract Prediction of atmospheric phenomena needs three components: a theoretical (or numerical) model based on the natural laws (physical, chemical, or biological), a sampling set of the reality, and a tolerance level. Comparison between the predicted and sampled values leads to the estimation of model error. In the error phase space, the prediction error is treated as a point; and the tolerance level (a prediction parameter) determines a tolerance-ellipsoid. The prediction continues until the error first exceeding the tolerance level (i.e., the error point first crossing the tolerance-ellipsoid), which is the first-passage time. Well-established theoretical framework such as backward Fokker-Planck equation can be used to estimate the first-passage time – an up time limit for any model prediction. A population dynamical system is used as an example to illustrate the concept and methodology and the dependence of the first-passage time on the model and prediction parameters.

1. Introduction

Global numerical weather prediction (NWP) started from the use of multi-level, primitive-equation model. For dry adiabatic atmosphere, the equations for momentum, thermodynamics, continuity, and the hydrostatic relation in σ-coordinate system (Phillips, 1959, 1970) are given by

$$\frac{D\mathbf{V}}{Dt} = -f\mathbf{k}\times\mathbf{V} - \nabla\Phi - RT\nabla \ln p_* + \mathbf{F}, \quad (1)$$

$$\frac{DT}{Dt} = \frac{RT}{c_p}\left(\frac{\dot{\sigma}}{\sigma} - \frac{\partial\dot{\sigma}}{\partial\sigma} - \nabla\bullet\mathbf{V}\right), \quad (2)$$

$$\frac{D\ln p_*}{Dt} = -\nabla\bullet\mathbf{V} - \frac{\partial\dot{\sigma}}{\partial\sigma}, \quad (3)$$

$$\frac{\partial\Phi}{\partial\sigma} = -\frac{RT}{\sigma}. \quad (4)$$

where the vertical coordinate is defined as $\sigma = p/p_*$ where p_* denotes the surface pressure and p the pressure within the atmosphere. **V** is the horizontal wind vector with eastward and northward components of u and v, respectively; T is the absolute temperature; Φ is the geopotential height; f is the Coriolis parameter; **k** is the vertical unit vector; ∇ is the horizontal gradient operator, $\dot{\sigma}$ is the total time derivative of σ, cp is the specific heat at constant pressure for dry air; and D/Dt is total time derivative.

Robert (1966) noted that the two scalars u and v are not well suited to representation in terms of scalar spectral expansions. Robert suggested that the variables,

$$U = u\cos\phi, \quad V = v\cos\phi, \quad (5)$$

would be more appropriate in the global spectral formulation. Here, ϕ is latitude. Use of the Helmholtz theorem, the velocity components are represented by a stream function (ψ) and a

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velocity potential (χ),

$$U = -\frac{\cos\phi}{a}\frac{\partial\psi}{\partial\phi} + \frac{1}{a}\frac{\partial\chi}{\partial\lambda}, \qquad (6)$$

$$V = \frac{1}{a}\frac{\partial\psi}{\partial\lambda} + \frac{\cos\phi}{a}\frac{\partial\chi}{\partial\phi}, \qquad (7)$$

where (a, λ, ϕ) are earth radius, longitude, and latitude.

The dependent variables are decomposed into spectral form with the basis functions of $Y_l^m(\phi,\lambda) = P_l^m(\sin\phi)e^{im\lambda}$, where $P_l^m(\sin\phi)$ is an associated Legendre polynomial of the first kind normalized to unity. Let the spectral amplitudes for the velocity potential, temperature, and surface pressure (logarithm) be represented by a vector **x,** which depends on time only. The spectral model is to predict the weather through solving the following ordinary differential equation for **x** (Bourke, 1974),

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t), \qquad (8)$$

where **f** is a functional representing the spectral amplitude tendencies.

When the mode number tends to infinity, the dynamical system (8) is the same as the original equations (1)-(4). Practically, the spectral model is truncated at a certain wave number *J*.

Uncertainty in spectral models (such as mode truncation, subgrid-scale paramterization, etc.) leads to the addition of stochastic forcing. For simplicity, a stochastic forcing (**f**′) is assumed to be white multiplicative or additive noise, and (8) becomes

$$\frac{d\hat{\mathbf{x}}}{dt} = \mathbf{f}(\hat{\mathbf{x}}, t) + \mathbf{f}'(\hat{\mathbf{x}}, t),$$

$$\mathbf{f}'(\hat{\mathbf{x}}, t) = \mathbf{k}(\hat{\mathbf{x}}, t)\mathbf{g}(t), \qquad (9)$$

where $\mathbf{k}(\mathbf{x}, t)$ and $\mathbf{g}(t)$ are the forcing covariance matrix $\{\mathbf{k}_{ij}\}$ (dimension of *J J*) and the vector delta-correlated process (dimension of *J*), respectively.

Let $\mathbf{x}(t)$ be the reference solution which satisfies (8) with the reference initial condition,

$$\mathbf{x}(t_0) = \mathbf{x}_0 .$$

Individual prediction using the spectral model is to integrate the differential equation (9) from the initial condition,

$$\hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0 .$$

The model error **z** is determined as

$$\mathbf{z}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t), \qquad (10)$$

Two vectors $\mathbf{x}(t)$ and $\hat{\mathbf{x}}$(t) are considered as reference and prediction points in the *J*-dimensional phase space.

A question arises: How long is the model (9) valid since being integrated from its initial state? This has great practical significance. In this paper, probabilistic stability analysis is proposed to investigate the model valid period. This method is on the base of the first-passage time (FPT) for model prediction.

2. First-Passage Time

In NWP, two model error limits should be priori defined. First, the forecast error cannot be less than a minimum scale δ, which depends on the intrinsic noises existing in the model. Second, the forecast error cannot be more than a maximum scale (tolerance level) ε. The prediction is valid if the reference point $\mathbf{x}(t)$ is situated inside the ellipsoid (S_ε, called tolerance ellipsoid) with center at the prediction point $\hat{\mathbf{x}}$ (*t*) and size ε. When $\mathbf{x}(t)$ coincides with $\hat{\mathbf{x}}$ (*t*), the model has perfect prediction. The prediction is invalid if the reference point $\mathbf{x}(t)$ touches the boundary of the tolerance ellipsoid at the first time from the initial state that is first passage time (FPT) for prediction (Fig. 1). FPT is a random variable when the model has stochastic forcing or initial condition has random error. Its statistics such as the probability density function, mean and variance can represent how long the model can predict. The FPT, $\tau = t - t_0$, depends on the initial model error, $\mathbf{z}_0 \equiv \mathbf{z}(t_0)$, tolerance level ε, and model parameters. The longer the FPT, the more stable of the economic model is Chu et. al., 2002a,b).

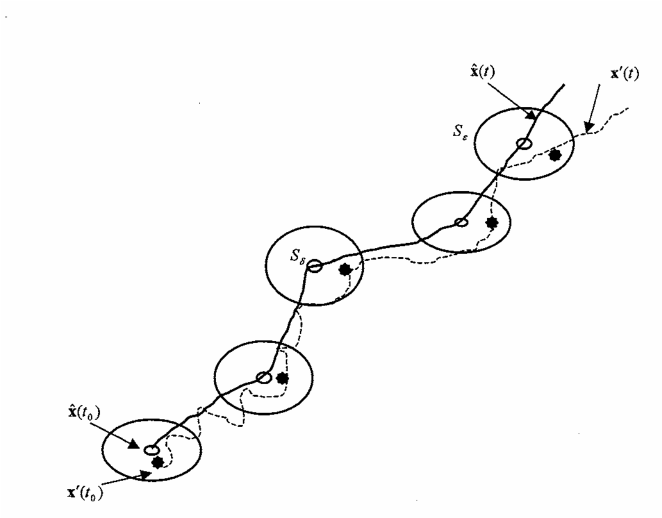


Figure 1. Phase space trajectories of model prediction $\hat{\mathbf{x}}$ (solid curve) and reality $\mathbf{x}$ (dashed curve) and error ellipsoid $S_\varepsilon(t)$ centered at $\hat{\mathbf{x}}$. The positions of reality and prediction trajectories at time instance are denoted by "*" and "o ", respectively). A valid prediction is represented by a time period $(t - t_0)$ at which the error first goes out of the ellipsoid $S_\varepsilon(t)$.

3. Backward Fokker-Planck Equation

The conditional probability density function (PDF) of FPT with a given initial error, $P[(t-t_0)\,|\,\mathbf{z}_0]$, satisfies the backward Fokker-Planck equation (Pontryagin et al. 1962; Gardiner, 1985; Chu et al. 2002)

$$\frac{\partial P}{\partial t} - \sum_{i=1}^{J}\left(f_i - \frac{d\hat{x}_i}{dt}\right)\frac{\partial P}{\partial z_i^0} - \frac{1}{2}\sum_{i=1}^{J}\sum_{j=1}^{J}\sum_{l=1}^{J} k_{il}k_{lj}\,\frac{\partial^2 P}{\partial z_i^0 \partial z_j^0} = 0, \tag{11}$$

where the coefficients k_{ij} are the components of the forcing covariance matrix $\boldsymbol{\kappa}(\mathbf{x},t)$ and $(z_1^0, z_2^0, \ldots z_J^0)$ are the components of the initial error $\mathbf{z}_0$. Integration of PDF over t leads to,

$$\int_{t_0}^{\infty} P[(t-t_0)\,|\,\mathbf{z}_0]\,dt = 1\,. \tag{12}$$

The k-th FPT moment (k = 1, 2, …) is calculated by,

$$\tau_k(\mathbf{z}_0) = k\int_{t_0}^{\infty} P[(t-t_0)\,|\,\mathbf{z}_0]\ \left(t\text{-}t_0\right)^{k\text{-}1} dt, \quad k = 1,\ldots,\infty\,. \tag{13}$$

If the initial error $\mathbf{z}_0$ reaches the tolerance level, the model loses prediction capability initially (i.e., FPT is zero)

$$P[(t-t_0)\,|\,\mathbf{z}_0] = 0, \qquad \text{at} \quad J(\mathbf{z}_0) = \varepsilon^2, \tag{14a}$$

which is the absorbing type boundary condition. Here $J(\mathbf{z}_0)$ denotes the norm of $\mathbf{z}_0$. If the initial error reaches the noise level the boundary condition becomes (Gardiner, 1985)

$$\frac{\partial P[(t-t_0)\,|\,\mathbf{z}_0]}{\partial z_0^{(j)}} = 0\,, \quad \text{at} \quad J(\mathbf{z}_0) = \delta^2\,, \tag{14b}$$

which is the reflecting boundary conditions. Here, ξ is the noise level. Usually,

$$\delta \ll \varepsilon\,.$$

Mean, variance, skewness, and kurtosis of the FPT are calculated from the first four moments,

$$\langle \tau \rangle = \tau_1\,, \tag{15a}$$

$$\langle \delta\tau^2 \rangle = \tau_2 - \tau_1^2\,, \tag{15b}$$

$$\langle \delta\tau^3 \rangle = \tau_3 - 3\tau_2\tau_1 + 2\tau_1^3\,, \tag{15c}$$

$$\langle \delta\tau^4 \rangle = \tau_4 - 4\tau_3\tau_1 + 6\tau_2\tau_1^2 - 3\tau_1^4\,, \tag{15d}$$

where the bracket denotes the ensemble average over realizations generated by stochastic forcing.

4. Lorenz System – A Simple Spectral Model

The Lorenz (1984) system is taken as an example to demonstrate the usefulness of the FPT approach. The Lorenz system is the simplest possible model capable of representing an unmodified or modified Hadley circulation, determining its stability, and, if it is unstable, representing a stationary or migratory disturbance. The model consists of the three ordinary nondimensional differential equations

$$\frac{dx_1}{dt} = -x_2^2 - x_3^2 - ax_1 + aF,$$
$$\frac{dx_2}{dt} = x_1x_2 - bx_1x_3 - x_2 + G, \qquad (16)$$
$$\frac{dx_3}{dt} = bx_1x_2 + x_1x_3 - x_3,$$

where t is the time scaled by five days; x_1 represents the intensity of the symmetric globe-encircling westerly wind, and also the poleward temperature gradient, which is assumed to be in permanent equilibrium with the wind; x_2 and x_3 are the cosine and sine phases, respectively, of a chain of superposed large-scale eddies, which transport heat at a rate proportional to the square of their amplitude, and transport no angular momentum at all. The terms, aF and G, represent symmetric and asymmetric thermal forcing. The terms, x_1x_2 and x_1x_3, describe amplification of eddies through interaction with the westerly currents. The displacement of eddies by the westerly wind is parameterized using the terms $-bx_1x_3$ and bx_1x_2 . The parameter a is the damping coefficient.

The traditional stability analysis on the Lorenz model (16) leads to three Lyapunov exponents. Among them, only one is positive. Projection of the three-dimensional forecast error vector onto the unstable manifold leads to a self-consistent model (Nicolis 1992)

$$\frac{d\xi}{dt} = (\sigma - g\xi)\xi + \nu(t)\xi, \ \xi\Big|_{t=t_0} = \xi_0, \ \xi \in [0,\infty), \qquad (17)$$

where ξ is non-dimensional amplitude of error, g is a non-negative, generally time-independent nonlinear parameter whose properties depend on the underlying attractor. Eq.(17) is written in Ito form. The tangent approximation of error growth leads to $g = 0$. The eigenvector ξ is associated with the positive Lyapunov exponent (σ). The term $\nu(t)\xi$ is a specially chosen stochastic forcing with zero mean and pulse-type variance

$$\langle \nu(t) \rangle = 0, \langle \nu(t)\nu(t') \rangle = q^2 \Delta(t - t'), \qquad (18)$$

where the bracket is defined as ensemble mean over realizations generated by the stochastic forcing, Δ is the Delta function, and q^2 is the intensity of attractor fluctuations modeled as multiplicative noise forcing.

Combination of free model parameters σ, g and q^2 affects the model prediction skill. Nicolis (1992) used the following values

$$\sigma = 0.64, \quad g = 0.3, \quad q^2 = 0.2. \qquad (19)$$

In reality the dynamical characteristics of the atmosphere vary considerably between different synoptic situations. Model forecast skill depends on various factors such as season, location, and boundary conditions. Evidence shows variability of forecast skill in operational models such as the European Center for Medium-Range Weather Forecasts (ECMWF) model. Therefore, the parameters σ, g and q^2 should be time and attractor dependent. This problem is not addressed here. However, in contrasting to Nicolis (1992) we assume

$$0.2 \le \sigma \le 0.64, \ g = 0.3, \ 0.01 \le q^2 \le 0.6, \qquad (20)$$

in this study. Since the Nicolis model is analytical, the parameter should be small and

$$z_1 \equiv \varepsilon / \delta = 10^5$$

is assumed. This should be satisfactory for the quality analysis of the prediction skill.

Use of the self-consistent model (17) has several explicit advantages. First, it allows analytical study on linear and nonlinear perspectives of forecast error. Second, the methodology developed by Nicolis (1992) can be generalized to more realistic atmospheric models. Third, model contains several dynamical regimes of forecast error behavior. Obviously, their analytical study may be useful for the interpretation of results obtained by large atmospheric models.

5. Mean and Variance of FPT

How long is the model (17) valid once being integrated from the initial state? Or what are the mean and variance of FPT of (17)? To answer these questions, we should first find the equations depicting the mean and variance of FPT for (17). Applying the theory described in Sections 2 and 3 to the model (17), the backward Fokker-Planck equation becomes,

$$\frac{\partial P}{\partial t} - [\sigma\xi_0 - g\xi_0^{\ 2}]\frac{\partial P}{\partial \xi_0} - \frac{1}{2}q^2\frac{\partial^2 P}{\partial \xi_0^{\ 2}} = 0\,, \qquad (21)$$

with the initial error (ξ_0) bounded by,

$$\xi_{noise} \le \xi_0 \le \varepsilon\,.$$

We multiply Eq.(21) by $(t - t_0)$ and $(t - t_0)^2$, then integrate with respect to t from t_0 to ∞, and obtain the first two moments of FPT equation

$$(\sigma\xi_0 - g\xi_0^2)\frac{d\tau_1}{d\xi_0} + \frac{q^2\xi_0^2}{2}\frac{d^2\tau_1}{d\xi_0^{\ 2}} = -1\,, \qquad (22)$$

$$(\sigma\xi_0 - g\xi_0^2)\frac{d\tau_2}{d\xi_0} + \frac{q^2\xi_0^2}{2}\frac{d^2\tau_2}{d\xi_0^{\ 2}} = -2\tau_1\,, \qquad (23)$$

with the boundary conditions,

$$\tau_1 = 0,\ \ \tau_2 = 0\,, \quad \text{for } \xi_0 = \varepsilon \qquad (24)$$

$$\frac{d\tau_1}{d\xi_0} = 0,\ \ \frac{d\tau_2}{d\xi_0} = 0\,, \text{ for } \xi_0 = \xi_{\text{noise}}. \qquad (25)$$

Analytical solutions of (22) and (23) with the boundary conditions (24) and (25) are

$$\tau_1(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon) = \frac{2}{q^2}\int_{\bar{\xi}_0}^{1} y^{-\frac{2\sigma}{q^2}} \exp\left(\frac{2\varepsilon g}{q^2}y\right)\left[\int_{\bar{\xi}_{noise}}^{y} x^{\frac{2\sigma}{q^2}-2}\exp\left(-\frac{2\varepsilon g}{q^2}x\right)dx\right]dy \qquad (26)$$

and

$$\tau_2(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon) = \frac{4}{q^2}\int_{\bar{\xi}_0}^{1} y^{-\frac{2\sigma}{q^2}} \exp\left(\frac{2\varepsilon g}{q^2}y\right)\left[\int_{\bar{\xi}_{noise}}^{y} \tau_1(x)x^{\frac{2\sigma}{q^2}-2}\exp\left(-\frac{2\varepsilon g}{q^2}x\right)dx\right]dy \qquad (27)$$

where

$$\bar{\xi}_0 = \xi_0/\varepsilon\,, \qquad \bar{\xi}_{noise} = \xi_{noise}/\varepsilon$$

are non-dimensional initial condition error and noise level scaled by the tolerance level ε, respectively. For given tolerance and noise levels (or user input), the mean and variance of FPT can be calculated using (26) and (27).

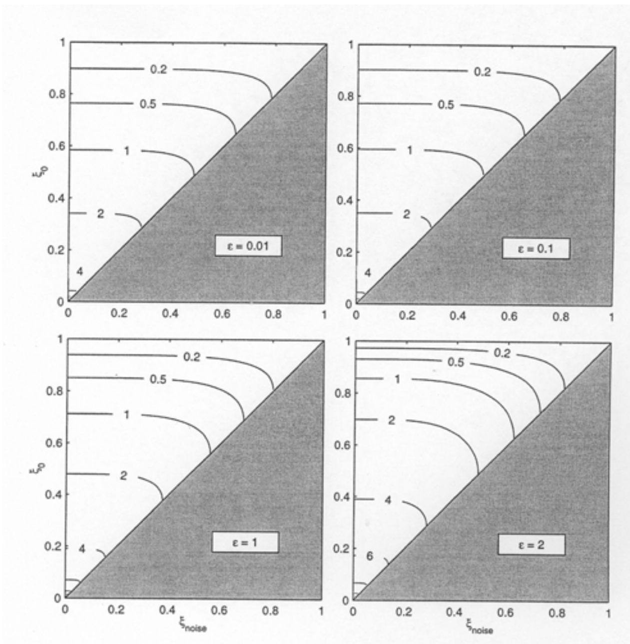
6. Dependence of τ_1 and τ_2 on $(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon)$

To investigate the sensitivity of τ_1 and τ_2 to $\bar{\xi}_0, \bar{\xi}_{noise}$, and , the same values are used for the parameters in the stochastic dynamical system (17) as in *Nicolis* (1992)

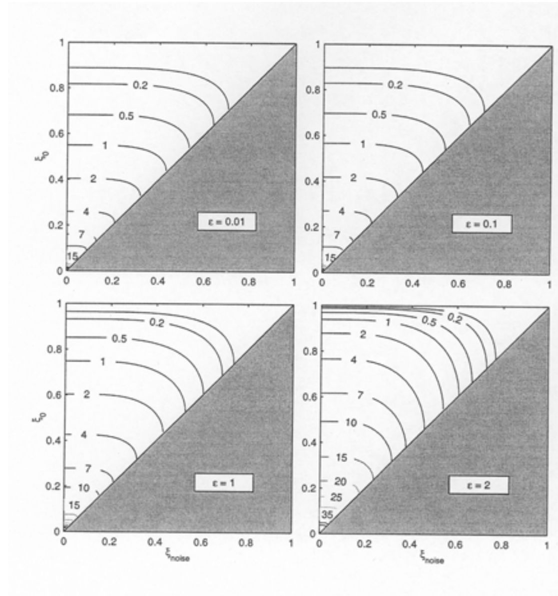
$$\sigma = 0.64,\ g = 0.3,\ q^2 = 0.2. \qquad (28)$$

Figures 2 and 3 show the contour plots of $\tau_1(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon)$ and $\tau_2(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon)$ versus $(\bar{\xi}_0, \bar{\xi}_{noise})$ for four different values of ε (0.01, 0.1, 1, and 2). Following features can be obtained: (a) For given values of $(\bar{\xi}_0, \bar{\xi}_{noise})$ [i.e., the same location in the contour plots], both τ_1 and τ_2 increase with the tolerance level ε. (b) For a given value of tolerance level ε, both τ_1 and τ_2 are almost independent on the noise level $\bar{\xi}_{noise}$ (contours are almost paralleling to the horizontal axis) when the initial error ($\bar{\xi}_0$) is much larger than the noise level ($\bar{\xi}_{noise}$). This indicates that the effect of the noise level ($\bar{\xi}_{noise}$) on τ_1 and τ_2 becomes evident only when the initial error ($\bar{\xi}_0$) is close to the noise level ($\bar{\xi}_{noise}$). (c) For given values of (ε, $\bar{\xi}_{noise}$), both τ_1 and τ_2 decrease with increasing initial error $\bar{\xi}_0$.

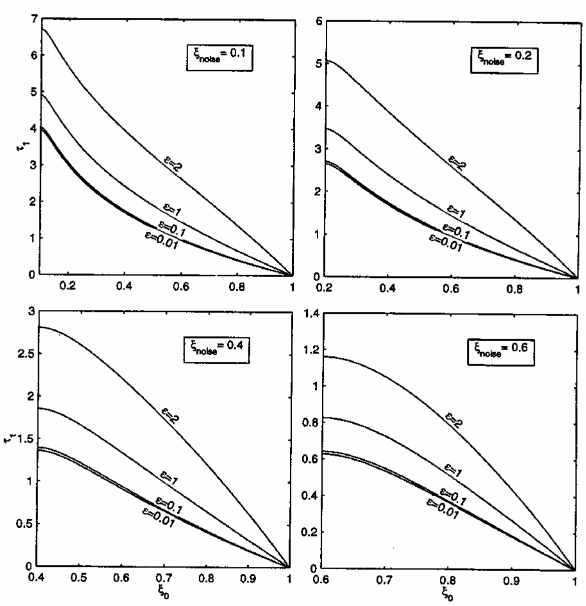
Figures 4 and 5 show the curve plots of $\tau_1(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon)$ and $\tau_2(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon)$ versus $\bar{\xi}_0$ for four different values of tolerance level, ε (0.01, 0.1, 1, and 2) and four different values of random noise $\bar{\xi}_{noise}$ (0.1, 0.2, 0.4, and 0.6). Following features are obtained: (a) τ_1 and τ_2 decrease with increasing $\bar{\xi}_0$, which implies that the higher the initial error, the lower the predictability (or FPT) is; (b) τ_1 and τ_2 decrease with increasing noise level $\bar{\xi}_{noise}$, which implies that the higher the noise level, the lower the predictability (or FPT) is; and (c) τ_1 and τ_2 increase with the increasing ε, which implies that the higher the tolerance level, the longer the FPT is. Note that the results presented in this subsection is for a given value of stochastic forcing (q^2 = 0.2) only.



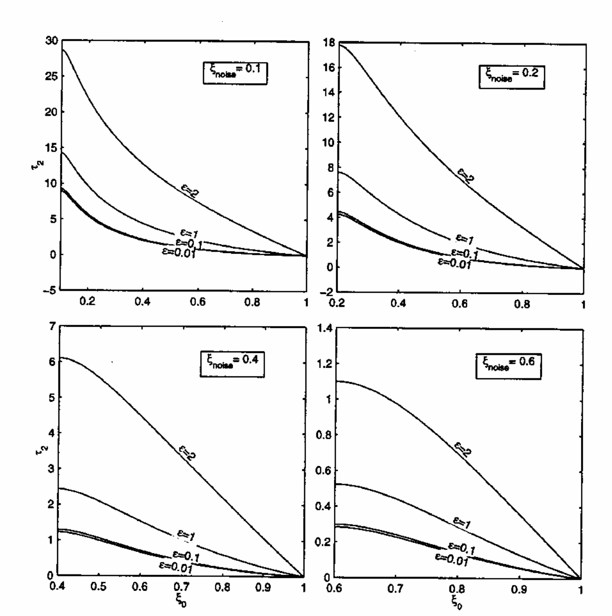
Figures 2. Contour plots of $\tau_1(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon)$ versus $(\bar{\xi}_0, \bar{\xi}_{noise})$ for four different values of ε (0.01, 0.1, 1, and 2) using Nicolis model with stochastic forcing q^2 = 0.2. The contour plot covers the half domain due to $\bar{\xi}_0 \geq \bar{\xi}_{noise}$.



Figures 3. Contour plots of $\tau_2(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon)$ versus $(\bar{\xi}_0, \bar{\xi}_{noise})$ for four different values of ε (0.01, 0.1, 1, and 2) using Nicolis model with stochastic forcing q^2 = 0.2. The contour plot covers the half domain due to $\bar{\xi}_0 \geq \bar{\xi}_{noise}$.



Figures 4. Dependence of $\tau_1(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon)$ on the initial condition error $\bar{\xi}_0$ for four different values of ε (0.01, 0.1, 1, and 2) and four different values of random noise $\bar{\xi}_{noise}$ (0.1, 0.2, 0.4, and 0.6) using Nicolis model with stochastic forcing q^2 = 0.2.

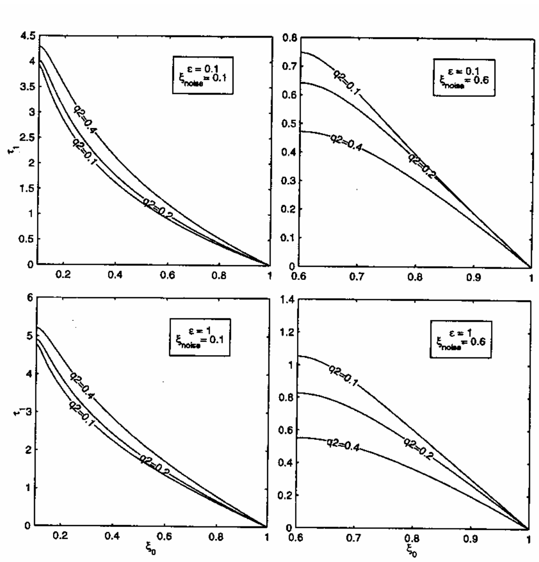


Figures 5. Dependence of $\tau_2(\overline{\xi}_0, \overline{\xi}_{noise}, \varepsilon)$ on the initial condition error $\overline{\xi}_0$ for four different values of ε (0.01, 0.1, 1, and 2) and four different values of random noise $\overline{\xi}_{noise}$ (0.1, 0.2, 0.4, and 0.6) using Nicolis model with stochastic forcing $q^2 = 0.2$.

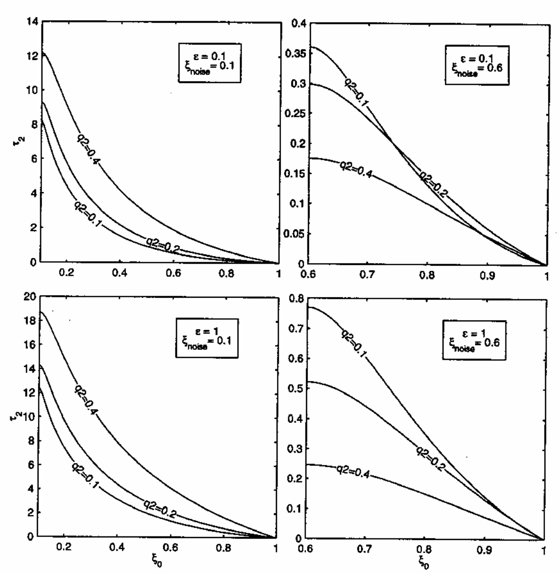
6. Dependence of τ_1 and τ_2 on Stochastic Forcing (q^2)

To investigate the sensitivity of τ_1 and τ_2 to the strength of the stochastic forcing, q^2, we use the same values for the parameters (σ = 0.64, g = 0.3) in (30) as in *Nicolis* [1992] except q^2, which takes values of 0.1, 0.2, and 0.4.

Figures 6 and 7 show the curve plots of $\tau_1(\overline{\xi}_0, \overline{\xi}_{noise}, q^2)$ and $\tau_2(\overline{\xi}_0, \overline{\xi}_{noise}, q^2)$ versus $\overline{\xi}_0$ for two tolerance levels (ε = 0.1, 1), two noise levels ($\overline{\xi}_{noise}$= 0.1, 0.6), and three different values of q^2 (0.1, 0.2, and 0.4) representing weak, normal, and strong stochastic forcing. Two regimes are found: (a) τ_1 and τ_2 decrease with increasing q^2 for large noise level ($\overline{\xi}_{noise}$= 0.6), (b) τ_1 and τ_2 increase with increasing q^2 for small noise level ($\overline{\xi}_{noise}$= 0.1), and (c) both relationships (increase and decrease of τ_1 and τ_2 with increasing q^2) are independent of ε.



Figures 6. Dependence of $\tau_1(\overline{\xi}_0, \overline{\xi}_{noise}, q^2)$ on the initial condition error $\overline{\xi}_0$ for three different values of the stochastic forcing q^2 (0.1, 0.2, and 0.4) using Nicolis model with two different values of ε (0.1, and 1) and two different values of noise level $\overline{\xi}_{noise}$ (0.1, and 0.6).



Figures 7. Dependence of $\tau_2(\overline{\xi}_0, \overline{\xi}_{noise}, q^2)$ on the initial condition error $\overline{\xi}_0$ for three different values of the stochastic forcing q^2 (0.1, 0.2, and 0.4) using Nicolis model with two different values of ε (0.1, and 1) and two different values of noise level $\overline{\xi}_{noise}$ (0.1, and 0.6).

This indicates the existence of stabilizing and destabilizing regimes of the dynamical system depending on stochastic forcing. For a small noise level, the stochastic forcing stabilizes the dynamical system and increase the mean FPT. For a large noise level, the stochastic forcing destabilizes the dynamical system and decreases the mean FPT.

The two regimes can be identified analytically for small tolerance level ($\varepsilon \to 0$). The initial error $\bar{\xi}_0$ should also be small ($\bar{\xi}_0$ ε). The solutions (26) becomes

$$\lim_{\varepsilon \to 0} \tau_1(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon) =$$

$$\frac{1}{\sigma - q^2/2}\left\{ \ln\left(\frac{1}{\bar{\xi}_0}\right) - \frac{q^2}{2\sigma - q^2}\bar{\xi}_{noise}^{\frac{2\sigma}{q^2}-1}\left[\left(\frac{1}{\bar{\xi}_0}\right)^{\frac{2\sigma}{q^2}-1} - 1\right]\right\} \tag{29}$$

The Lyapunov exponent is identified as (σ - $q^2/2$) for dynamical system (17) [*Hasmin'skii* 1980].

For a small noise level ($\bar{\xi}_{noise}$ 1), the second term in the bracket of the righthand of (29)

$$R = -\frac{q^2}{2\sigma - q^2}\bar{\xi}_{noise}^{\frac{2\sigma}{q^2}-1}\left[\left(\frac{1}{\bar{\xi}_0}\right)^{\frac{2\sigma}{q^2}-1} - 1\right], \tag{30}$$

is negligible. The solution (29) becomes

$$\lim_{\varepsilon \to 0} \tau_1(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon) = \frac{1}{\sigma - q^2/2}\ln\left(\frac{1}{\bar{\xi}_0}\right), \tag{31}$$

which shows that the stochastic forcing ($q \neq 0$), reduces the Lyapunov exponent (σ - $q^2/2$), stabilizes the dynamical system (17), and in turn increases the mean FPT. On the other hand, the initial error $\bar{\xi}_0$ reduces the mean FPT.

For a large noise level $\bar{\xi}_{noise}$, the second term in the bracket of the righthand of (29) is not negligible. For a positive Lyapunov exponent, $2\sigma - q^2 > 0$, this term is always negative [see (30)]. The absolute value of R increases with increasing q^2 (remember that $\bar{\xi}_{noise} < 1$, $\bar{\xi}_0 < 1$). Thus, the term (R) destabilizes the one-dimensional stochastic dynamical system (17), and reduces the mean FPT.

7. Conclusions

(1) FPT (a single scalar) represents the model predictability skill. It depends not only on the instantaneous error growth, but also on the noise level, the tolerance level, and the initial error. A theoretical framework was developed in this study to determine the mean (τ_1) and variability (τ_2) of valid prediction period for nonlinear stochastic dynamical system. The probability density function of the valid prediction period satisfies the backward Fokker-Planck equation. After this equation, it is easy to obtain the ensemble mean and variance of the valid prediction period.

(2) Uncertainty in atmospheric models are caused by measurement errors (initial and/or boundary condition errors), model discretization, and uncertain model parameters. This motivates to the inclusion of stochastic forcing in ocean (atmospheric) models. The backward Fokker-Planck equation can be used for evaluation of ocean (or atmospheric) model predictability through calculating the mean FPT.

(3) For the Nicolis (1992) model, the second-order ordinary differential equations of τ_1 and τ_2 have analytical solutions, which clearly show the following features: (a) decrease of τ_1 and τ_2 with increasing initial condition error (or with increasing random noise), (b) increase of τ_1 and τ_2 with increasing tolerance level ε.

(4) Both stabilizing and destabilizing regimes are found in the Nicolis model depending on stochastic forcing. For a small noise level, the stochastic forcing stabilizes the dynamical system and increases the mean VPP. For a large noise level, the stochastic forcing destabilizes the dynamical system and decreases the mean VPP.

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