PAPER

The S-transform for Obtaining Localized Spectra

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ABSTRACT

We used the recently developed S-transform, an extention to the ideas of the Gabor transform and wavelet transform, to analyze the Tropical Ocean Global Atmosphere (TOGA) sea level data set and to obtain the localized spectra, varying with time. Our S-spectra show some features that cannot be obtained from the Fourier transform, such as phase lock, temporal variability of spectra, and out-of-phase behavior of the semiannual, annual, and biannual signals between the western Pacific (Naura) station and the eastern Pacific (La Libertad) station. The annual signal is usually phase-locked with the semiannual signal, but not with the biannual signal. The annual signal is quite stationary in the western Pacific station and non-stationary in the eastern Pacific station. During 1980-84, the quasi-biannual (QB) signal was very strong at the western Pacific station and quite weak at the eastern Pacific station. However, during 1974-76 and 1986-90, the QB signal was weak at the western Pacific station and strong at the eastern Pacific station. This may imply different physical processes involved in the western and eastern Pacific during the El Niño and Southern Oscillation (ENSO) periods.

INTRODUCTION

After Pedlosky (1970) proposed the dual time scale concept for nonlinear waves, a one-dimensional perturbation along x-axis is generally depicted as

$$\varphi = A(\omega, \tau)\, cos(kx - 2\pi\omega t) \qquad (1)$$

where k is the wave number, ω the frequency, and (τ, t) the dual time scales. The temporal variable t represents the short time scale (fast variation), and τ denotes the long time scale (slow variation). A is the amplitude which varies on a slower time scale τ.

Many papers have been published since then to determine $A(\omega, \tau)$ from different dynamical models, especially from finite-amplitude wave models (e.g., Pierrehumbert 1984; Pedlosky 1992). However, little work has been done to obtain $A(\omega, \tau)$ from real ocean data. The Fourier transform is more commonly used to obtain the spectrum from ocean data. The mutual relation of the Fourier spectrum $H(\omega)$ and its time series $h(t)$ is given by

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi\omega t}dt \qquad (2)$$

and

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{i2\pi\omega t}d\omega \qquad (3)$$

Obviously, the Fourier spectrum $H(\omega)$ does not depend on time. Statistically the amplitude $A(\omega, \tau)$ in (1) is considered as the absolute value of a localizing spectrum near time τ. To analyze the time variation of the sea surface height spectra, we need to use some other transform, such as a wavelet transform, the Short Time Fourier transform (i.e., the Gabor transform), or the S-transform.

The Gabor transform is given by (Gabor 1946)

$$\Gamma(\omega, \tau) = \int_{-\infty}^{\infty} h(t)g(t - \tau, \sigma)e^{-i2\pi\omega t}dt \quad (4)$$

where $g(t, \sigma)$ is called the Gaussian Window, defined by

$$g(t, \sigma) = \frac{1}{\sqrt{2\pi}\sigma}\, exp\left[-\left(\frac{t}{\sqrt{2}\sigma}\right)^2\right] \qquad (5)$$

where τ is the translation parameter with the same dimension as t, and σ is the window width. For any τ, the Gabor transform provides the spectrum of a windowed segment of $h(t)$, centered at time τ. The Gabor transform has two kinds of limitations. First, if the data set has a transient component with a scale smaller than σ, it is difficult to locate it with precision better than σ. Second, if the data set has important features of differing sizes then an optimal $g(t,\sigma)$ for analysis cannot be found. Therefore, the Gabor transform is more suitable for analyzing data where all features appear approximately at the same scale.

The wavelet transform $W(\tau, \sigma)$ addresses the resolution problem by introducing a dilation (or scale) parameter σ and is defined by

$$W(\tau, \sigma) = \int_{-\infty}^{\infty} h(t)w(t - \tau, \sigma)dt \qquad (6)$$

where the functions $w(t, \sigma)$ are called wavelets. The spectrum of $h(t)$ is obtained through correlation or convolution with $w(t, \sigma)$. The dilution scale σ determines the width of the wavelet $w(t, \sigma)$ and thus controls the resolution. In the wavelet transform, the frequency is not explicitly presented. There is no direct relationship between Fourier spectrum and wavelet spectrum.

Different from the Gabor transform and wavelet transform, the S-transform has variable windows and a close connection to the Fourier transform. In this paper, we use the S-transform to obtain a localized spectrum of the sea-level from the Tropical Ocean Global Atmosphere (TOGA) Data Set.

THE S-TRANSFORM

The S-transform, recently proposed by Stockwell et al. (1994), is a generalization of the Gabor transform, and an extension of the Wavelet transform. The mutual relation of the S-spectrum $S(\omega, \tau)$ and its time series $h(t)$is given by

$$S(\omega, \tau) = \int_{-\infty}^{\infty} H(\omega + \alpha) e^{-\frac{2\pi^2\alpha^2}{\omega^2}} e^{i2\pi\alpha\tau} d\alpha \quad (7)$$

and

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(\omega, \tau) e^{i2\pi\omega t} d\tau d\omega \quad (8)$$

where $H(\omega + \alpha)$ is the frequency-shifted Fourier spectrum of $h(t)$. Comparing Equation (8) with (3) one can find that the Fourier spectrum $H(\omega)$ is a time average of the S-spectrum $S(\omega, \tau)$:

$$H(\omega) = \int_{-\infty}^{\infty} S(\omega, \tau) d\tau \quad (9)$$

Equation (8) can also be viewed as the decomposition of a time series $h(t)$ into sinusoidal oscillations on time t with the temporal varying amplitudes $S(\omega, \tau)$.

Let $h(k\Delta T)$, $k = 0, 1, ..., N - 1$ denote time series, corresponding to $h(t)$, with a time sampling interval of ΔT. The discrete form of Equation (7) is given by

$$S\left[\frac{n}{N\Delta T}, j\Delta T\right] = \sum_{m=0}^{N-1} H\left(\frac{m+n}{N\Delta T}\right) e^{-\frac{2\pi^2 m^2}{n^2}} e^{\frac{i2\pi mj}{N}} \quad (10)$$

for $n \neq 0$ and by

$$S[0, j\Delta T] = \frac{1}{N} \sum_{m=0}^{N-1} H\left(\frac{m}{N\Delta T}\right) \quad (11)$$

for $n = 0$. Here the discrete Fourier spectrum is computed by (Brigham 1974):

$$H\left[\frac{n}{N\Delta T}\right] = \frac{1}{N} \sum_{m=0}^{N-1} h(k\Delta T)^{-\frac{i2\pi nk}{N}} \quad (12)$$

LIMITATION OF THE FOURIER TRANSFORM

The spectrum computed from the Fourier transform only shows overall behavior of a data set. Different data sets might have the same Fourier transform. For example, consider the following two signals:

$$h_1(t) = \sin(10t) + \sin(20t), \ 0 \le t \le 200$$

and

$$h_2(t) = 2\sin(20t), \ 0 \le t < 100$$
$$h_2(100) = \sin(2000) + \sin(1000)$$
$$h_2(t) = 2\sin(10t), \ 100 < t \le 200$$

The first signal consists of superposition of two frequencies (Figure 1a), and the second signal consists of the same two frequencies, each appearing separately over half of the signal duration (Figure 1b). The magnitudes of the two Fourier spectra are identical (Figure 1c, d), indicating the incapability of distinguishing the two signals. On the other hand, the magnitude of the two S-spectra are very different (Figure 1e, f).

TOGA SEA LEVEL DATA

The monthly sea level data provided by the TOGA Sea Level Center at the University of Hawaii is used to demonstrate the usefulness of the S-transform. For simplicity, only two stations for the computation are selected: Nauru 1974-90 (0°32′*S*, 166°54′*E*) and La Libertad 1971-89 (2°12′*S*, 80°55′*W*) representing equatorial western and eastern Pacific (Figure 2).

There are some missing observations in the two stations (Figure 3). Complete time series (no-missing data) for computing the spectrum is needed. Therefore, the linear interpolation to obtain values for the missing data is used. From the modified time series of sea-level (Figure 4), annual and interannual variabilities are evident at both stations.

FOURIER SPECTRA OF SEA LEVEL DATA

The sea level data have been detrended and nondimensionalized by

$$\eta = \frac{h - \bar{h}}{\bar{h}}$$

before computing the spectra. Here $\bar{h}$ is the mean value of the sea-level at the station. Due to the short time series, the Fourier spectra (Figure 5), $|H(\omega)|$, are useless for periods beyond sixty months. All the interesting peaks are crammed into the domain with periods shorter than thirty months: (1) several signals (semi-annual, annual, and bi-annual) appearing in both Nauru and La Libertad spectra; (2) a strong annual signal in the western Pacific; and (3) a strong bi-annual signal in the eastern Pacific. Thus, the Fourier spectra provides overall information (in the sense of time averaged spectrum) about the sea level variability. Do these signals (semi-annual, annual, and bi-annual) have constant amplitudes during the whole sampling period? To answer this question, we should use the S-transform.

Figure 1. Difference between the Fourier and S-transforms: (a) time series of h1, (b) time series of h2, (c) magnitude of the Fourier spectrum for h1, (d) magnitude of the Fourier spectrum for h2, (e) magnitude of the S-spectrum for h1, and (f) magnitude of the h2.

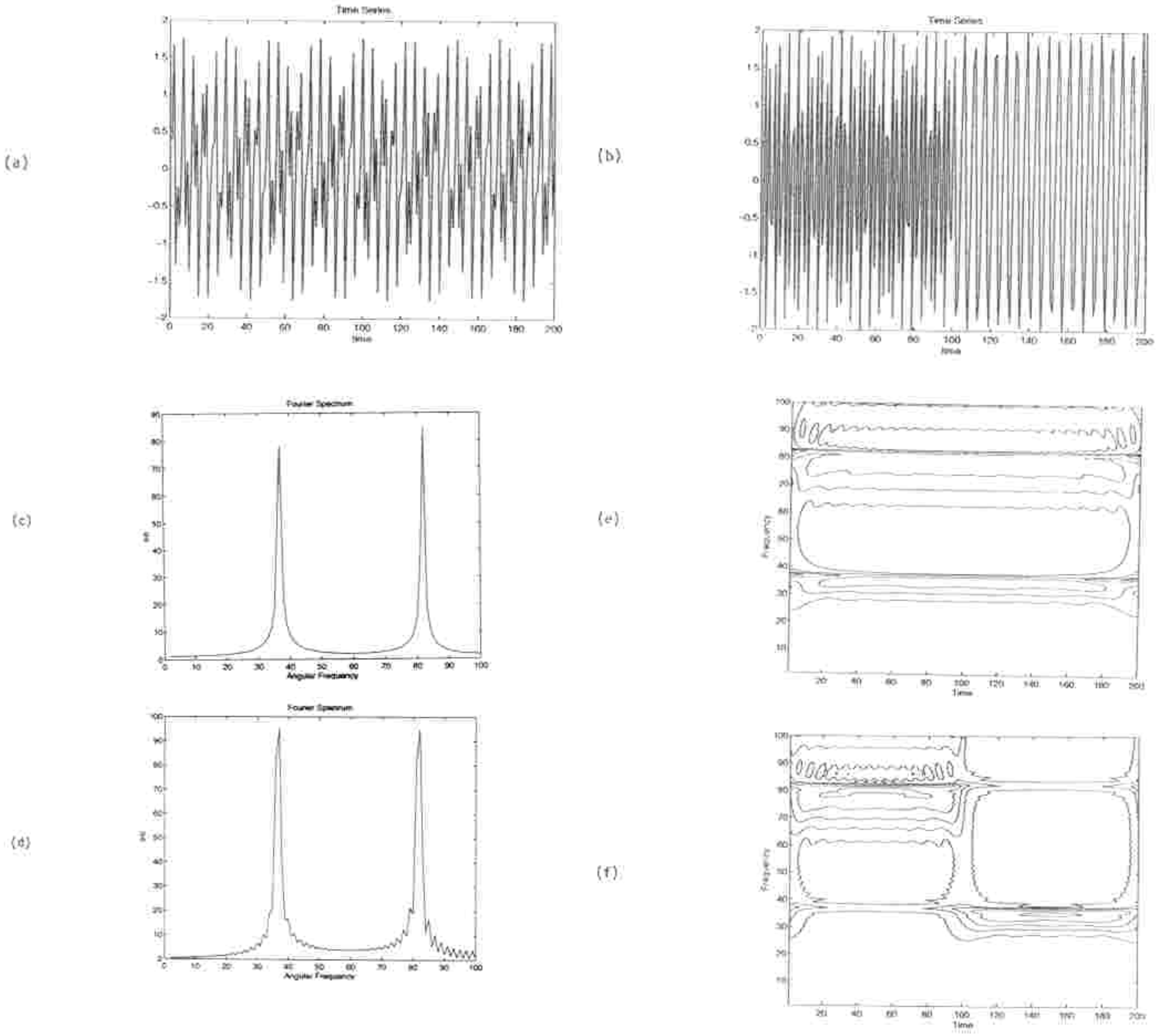
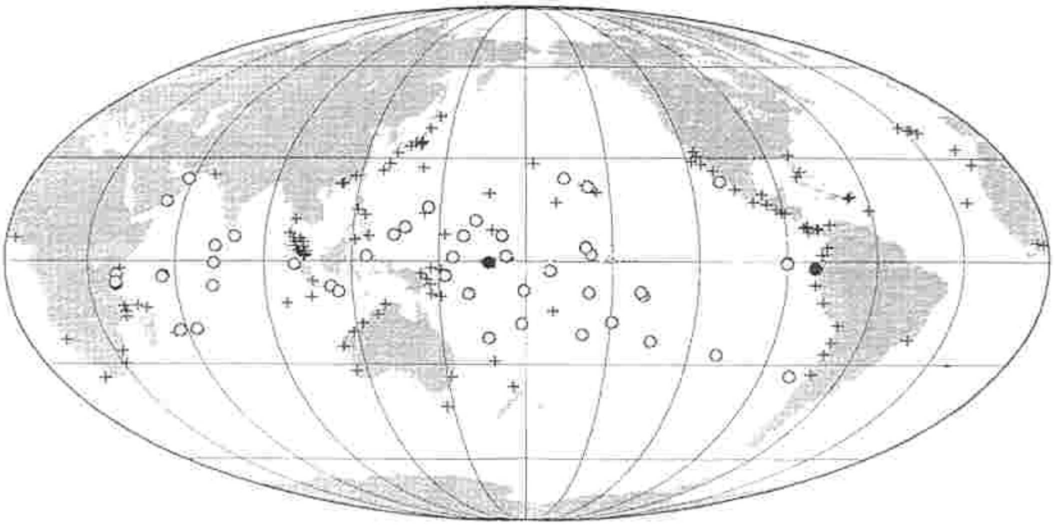


Figure 2. The TOGA sea level stations (Nauru and La Libertad stations are indicated by black dots).



S-SPECTRA OF SEA LEVEL DATA

Using Equations (10) and (11) we compute the S-spectra from the nondimensional sea level data for the two stations (Figure 2). S is a complex function of $\omega(=n/N\Delta T)$ and time $\tau(=j\Delta T)$. Here $\Delta T = 1$ month. The period P is defined by

$$P = \frac{1}{\omega}. \qquad (13)$$

If we use the unit of year for period, Equation (13) becomes

$$P = \frac{N}{12n} \qquad (14)$$

Thus the complex function S can be transformed into a function of P and τ. The amplitude and the phase of $S(P, \tau)$ are computed by

$$A(P, \tau) = |S(P, \tau)|, \qquad (15)$$
$$\Phi(P, \tau) = \arctan \frac{Im[S(P, \tau)]}{Re[S(P, \tau)]}.$$

The function A, called the S-spectrum, is plotted against the ($\log_2 P$) and time τ for Nauru and La Libertad, as shown in Figure 6. If we average $A(P, \tau)$ along the τ-axis, we will get the Fourier spectra. Figure 6 shows several interesting features at both stations: (1) the annual signal was phase locked with the semiannual signal in both the western and eastern Pacific since the maximum values of $A(1, \tau)$ usually correspond to the maximum values of $A(1/2, \tau)$; and (2) the phase lock of the annual and bi-annual signals is not very evident except during 1974-76 in the western Pacific (Figure 6a) where high values of $A(P, \tau)$ extend from $P = 1/2$ year to $P = 1$ year.

Figure 3. Original monthly sea level (mm) at (a) Nauru (western Pacific), and (b) La Libertad (eastern Pacific).

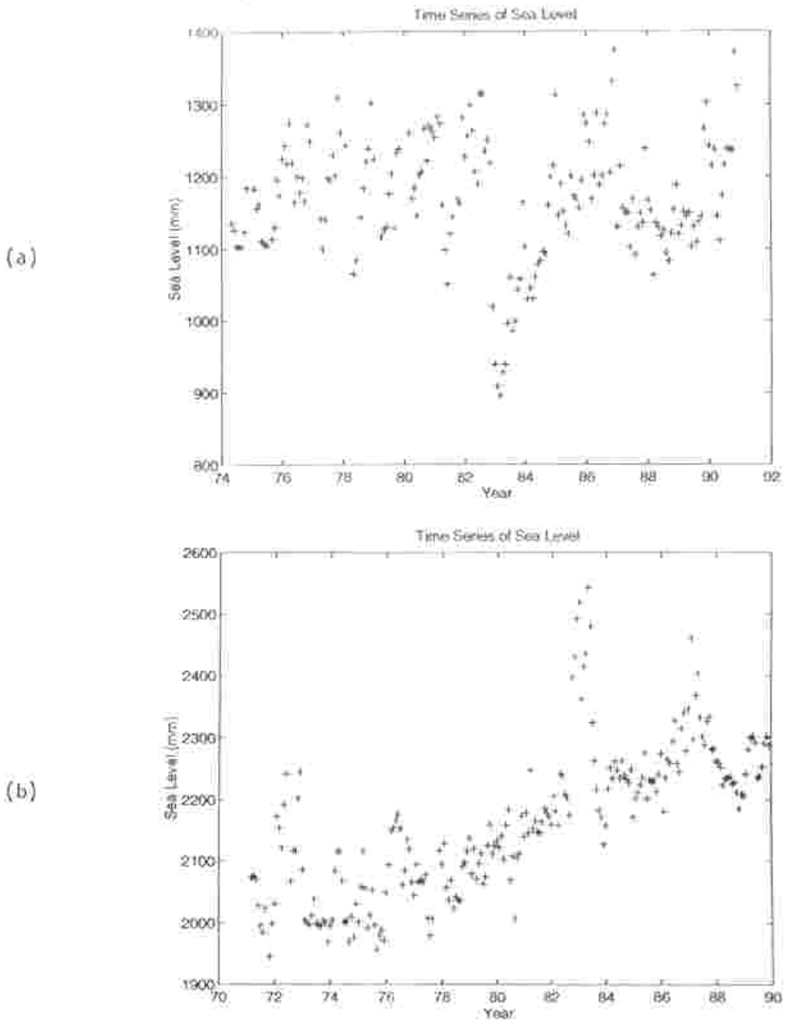
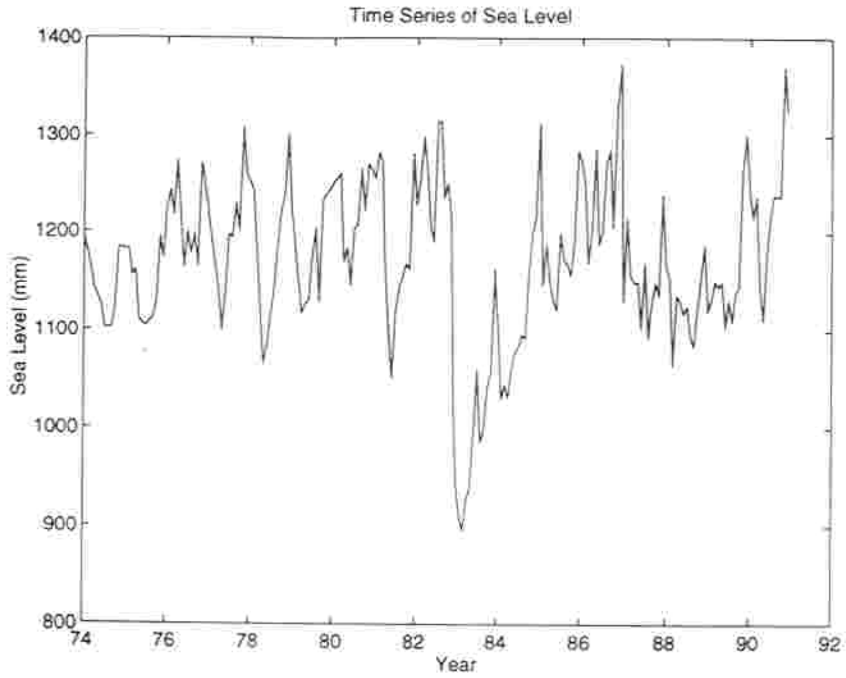


Figure 4. Modified monthly sea level (mm) at (a) Nauru (western Pacific), and (b) La Libertad (eastern Pacific).

(a)



(b)

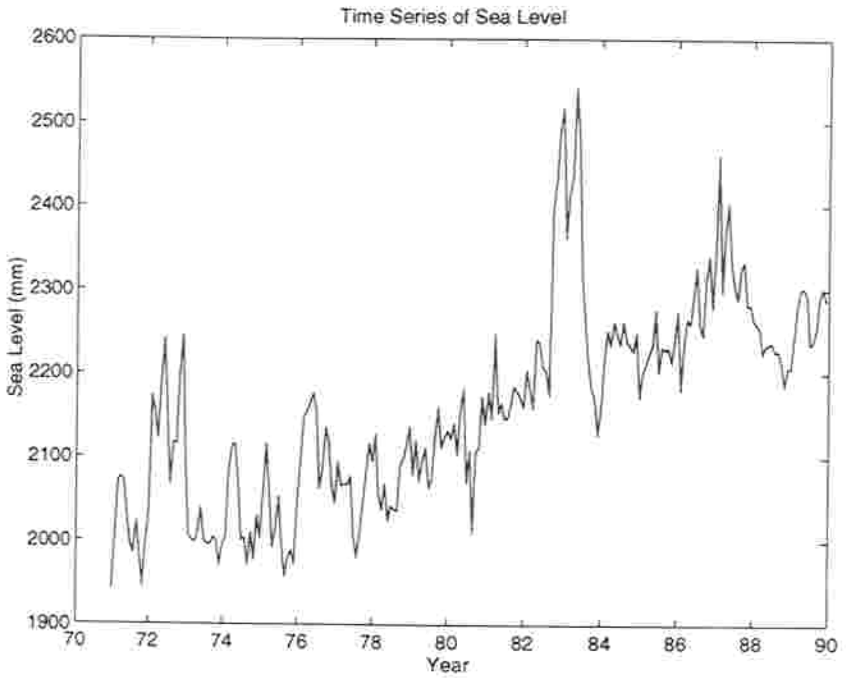


Figure 5. Fourier spectra of the sea level at (a) Nauru, and (b) La Libertad.

(a)

(b)

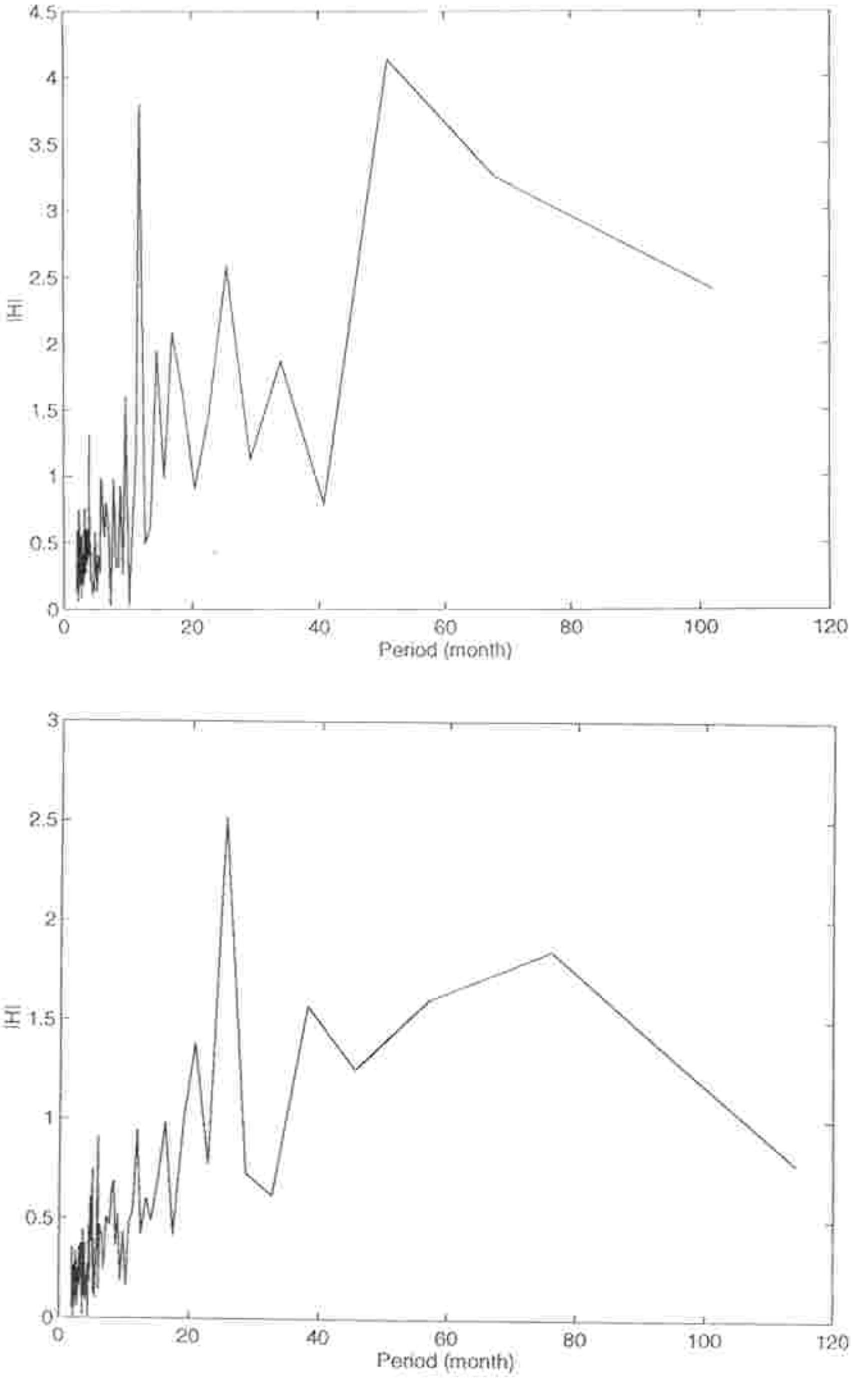
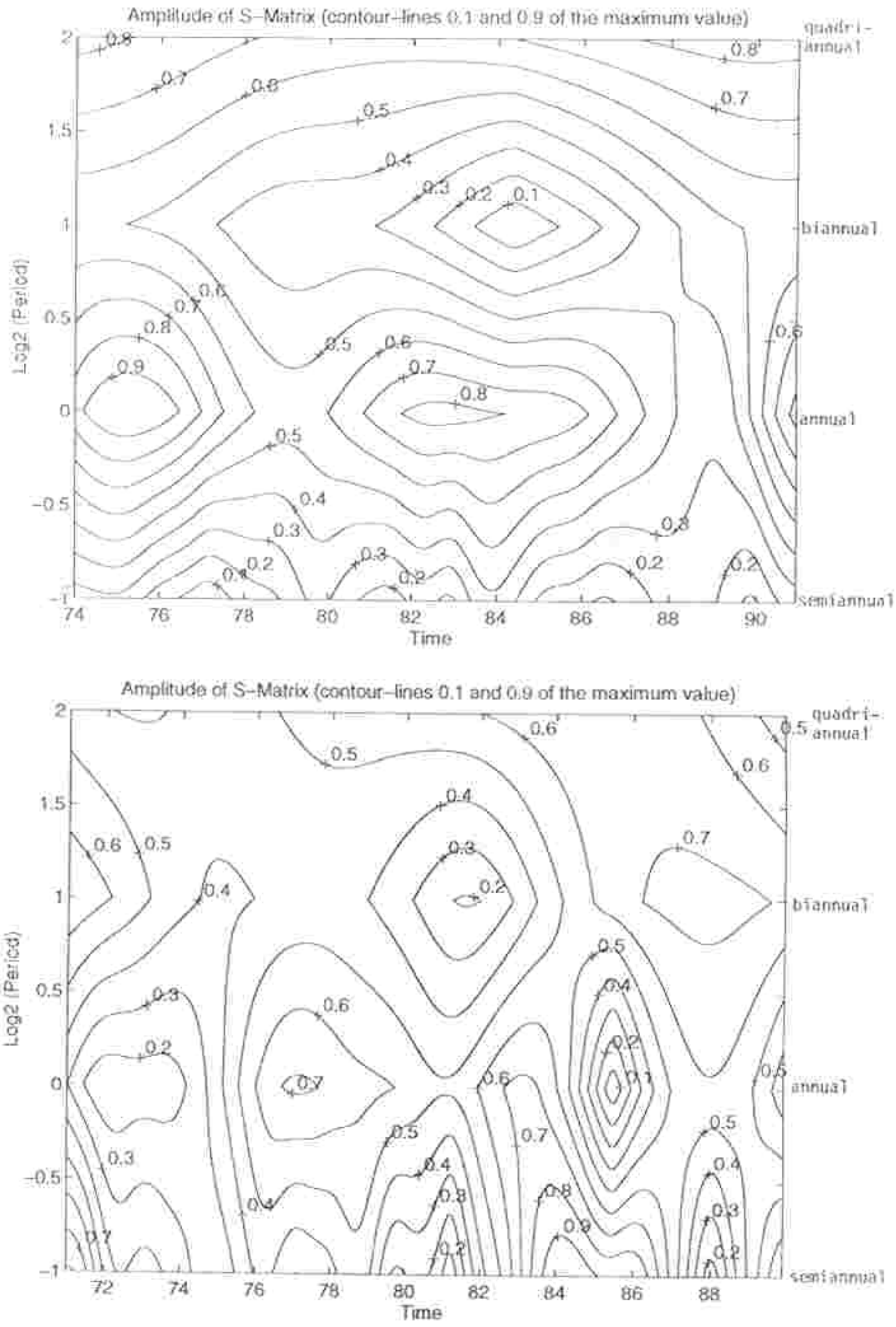


Figure 6. S-spectra normalized by the maximum value at (a) Nauru (maximum value = 5.59), and (b) La Libertad (maximum value = 2.36).

(a)

(b)



SEMIANNUAL AND ANNUAL SIGNALS

Since $A(P, \tau)$ and $\Phi(P, \tau)$ are the amplitude and phase of a complex variable, $S(P, \tau)$, the sinusoidal function

$$V(P, \tau) = A(P, \tau)^* \cos\left[\frac{2\pi\tau}{P} + \Phi(P, \tau)\right] \qquad (16)$$

provides quite useful information about the time evolution of the S-spectrum. The function, $V(P, \tau)$, evaluated at a particular period P, is called the "voice." The voice for $P = 1/2$ year (semi-annual signals) shows some interesting features at the western and eastern Pacific (Figure 7): (1) the weakest semiannual signal is found in 1977 in the western Pacific and in 1981 in the eastern Pacific; (2) the strongest semiannual signal is found in 1984 in both the western and eastern Pacific; and (3) during 1974-80 the strength of the semiannual signal varies rapidly in the western Pacific, but is quasi-stationary in the eastern Pacific.

The voice for $P = 1$ year (annual signals) indicates that (Figure 8): (1) the annual signal is quite stationary in the western Pacific and non-stationary in the eastern Pacific; and (2) during 1971-74 and 1984-86, the annual signal is very weak in the eastern Pacific.

QUASI-BIANNUAL (QB) SIGNALS

When the voice is evaluated at particular frequencies, $\omega_8 = 8/N\Delta T$, the function $V(\omega_8, \tau)$ represents QB signals: 25.5 month period at Nauru (Figure 9a) and 28.5 month period at La Libertad (Figure 9b). The QB signal has an out-of-phase feature at the western and eastern Pacific: large amplitudes appear during 1980-84 at Nauru and during 1972-73 and 1986-89 at La Libertad, and small amplitudes emerge during 1974-77 and 1987-90 at Nauru and during 1977-82 at La Libertad. This may imply different physical processes involved in the western and eastern Pacific during the El Niño and Southern Oscillation (ENSO) periods.

CONCLUSIONS

(1) Our S-spectra show some features that cannot be obtained from the Fourier transform, such as phase lock, temporal variability of spectra, and out-of-phase behavior of the semiannual, annual, and biannual signals between the western and eastern Pacific stations.

(2) The annual signal is usually phase-locked with the semiannual signal, but not with the biannual signal. The annual signal is quite stationary in the western Pacific station and

Figure 7. Voices of the semiannual- oscillation for (a) Nauru, and (b) La Libertad.

(a)

(b)

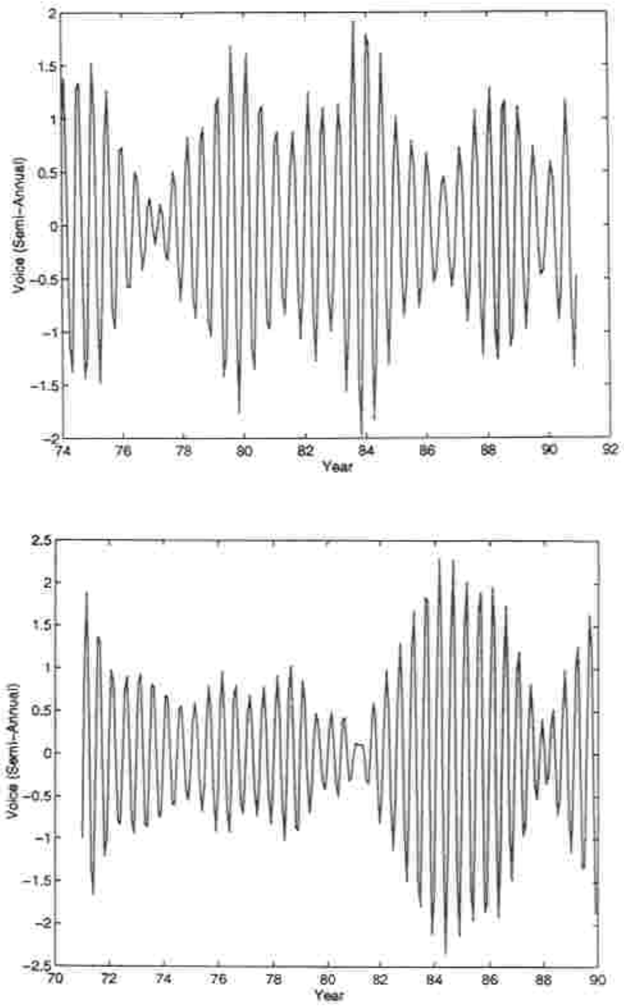
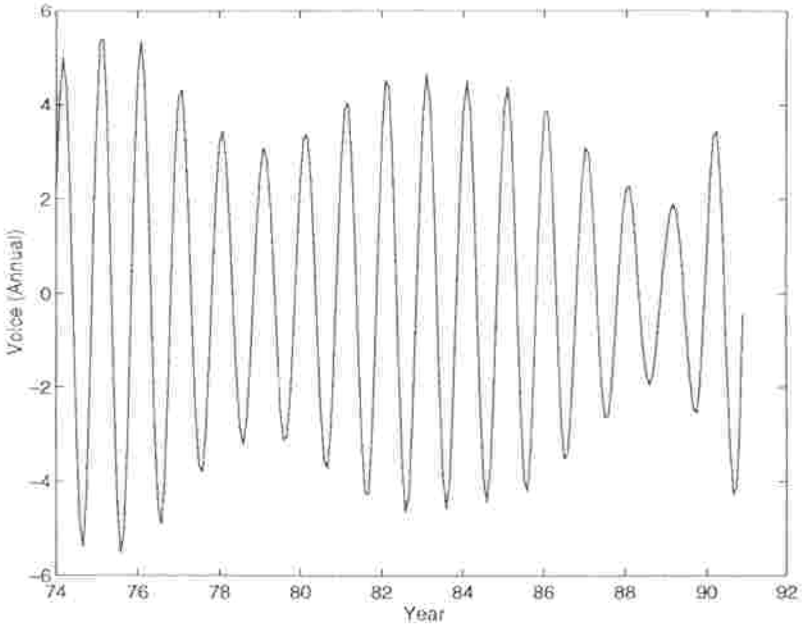


Figure 8. Voices of annual cycles for (a) Nauru, and (b) La Libertad.

(a)



(b)

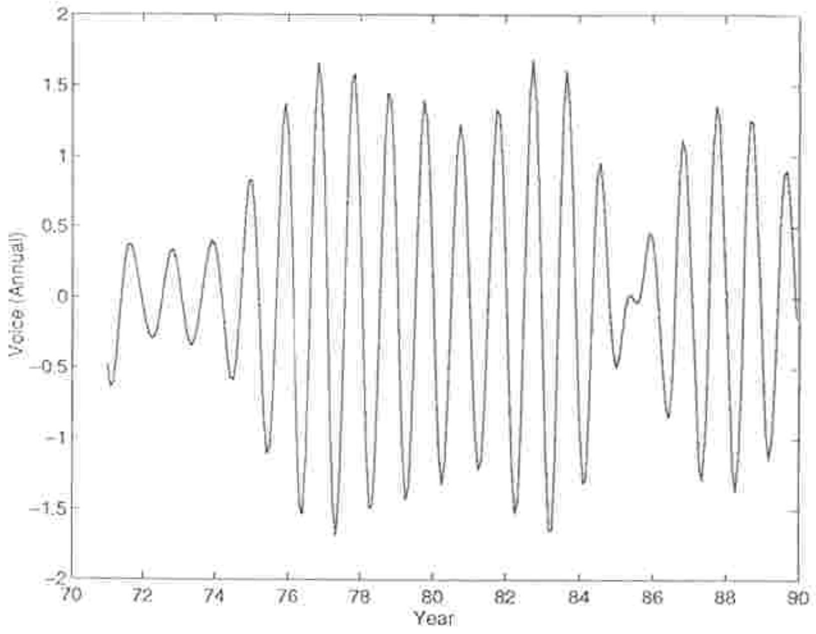
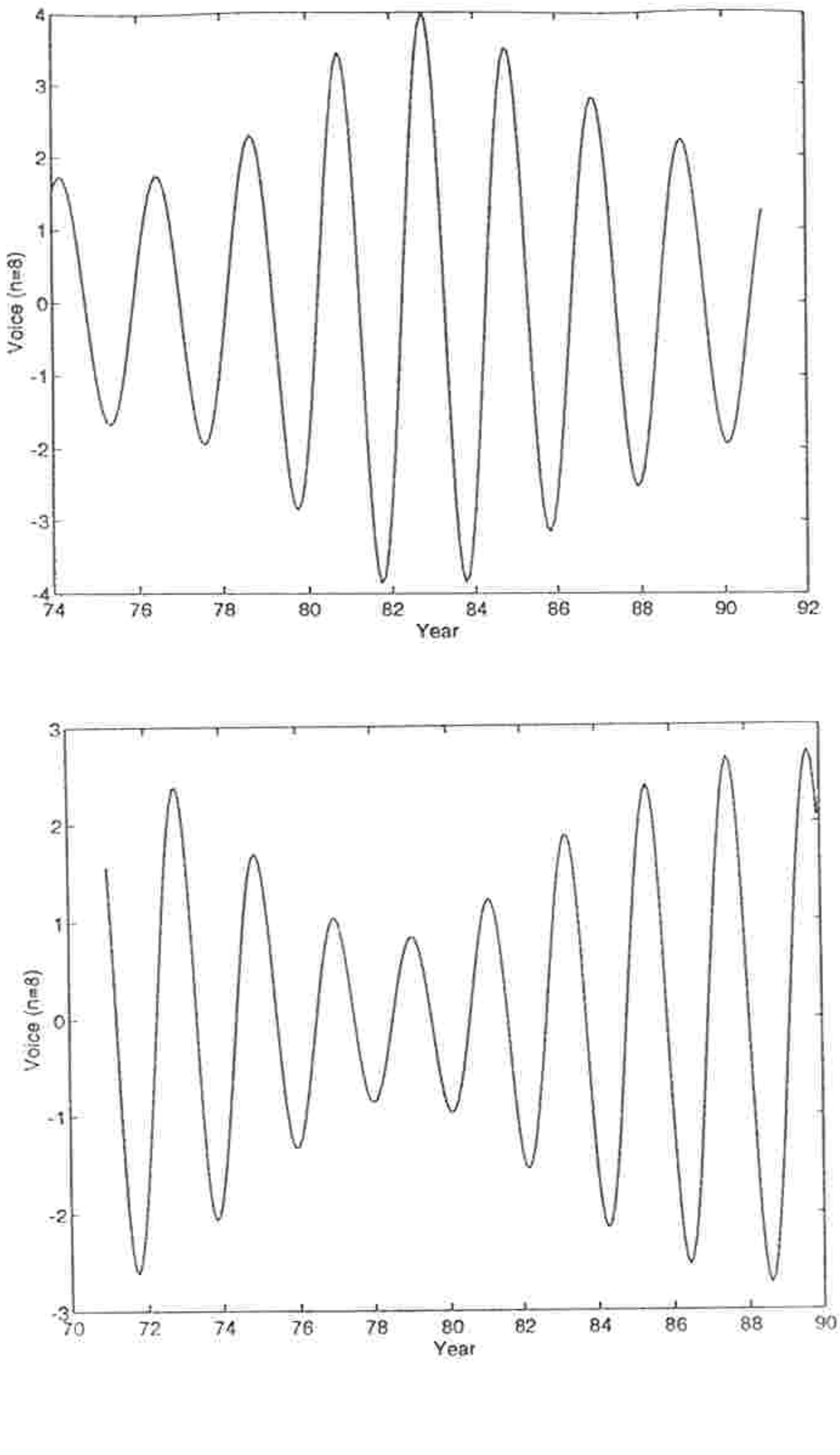


Figure 9. Voices of the quasi-biennual oscillation for (a) Nauru, and (b) La Libertad.

(a)

(b)



non-stationary in the eastern Pacific station.

(3) We found out-of-phase behavior of the QB signal in the western and eastern Pacific stations. During 1980-84, the QB signal is very strong at the western Pacific station and quite weak at the eastern Pacific station. However, during 1974-76 and 1986-90, the QB signal is weak at the western Pacific station and strong at the eastern Pacific station. This may imply different physical processes involved in the western and eastern Pacific during the ENSO periods.

(4) The statistical significance and confidence interval of the S-spectrum will be discussed in another paper.

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