Backward Fokker-Planck equation for determining model valid prediction period

Peter C. Chu, Leonid M. Ivanov, and Chenwu Fan
Department of Oceanography, Naval Postgraduate School, Monterey, California, USA

Received 15 March 2001; revised 17 December 2001; accepted 21 December 2001; published 26 June 2002.

[1] A new concept, valid prediction period (VPP), is presented here to evaluate ocean (or atmospheric) model predictability. VPP is defined as the time period when the prediction error first exceeds a predetermined criterion (i.e., the tolerance level). It depends not only on the instantaneous error growth but also on the noise level, the initial error, and the tolerance level. The model predictability skill is then represented by a single scalar, VPP. The longer the VPP, the higher the model predictability skill is. A theoretical framework on the basis of the backward Fokker-Planck equation is developed to determine the mean and variance of VPP. A one-dimensional stochastic dynamical system [*Nicolis*, 1992] is taken as an example to illustrate the benefits of using VPP for model evaluation. *INDEX TERMS:* 4263 Oceanography: General: Ocean prediction; 4255 Oceanography: General: Numerical modeling; 3367 Meteorology and Atmospheric Dynamics: Theoretical modeling; *KEYWORDS:* backward Fokker-Planck equation, instantaneous error, Lorenz system, predictability, tolerance level, valid prediction period

1. Introduction

[2] A practical question is commonly asked: How long is an ocean (or atmospheric) model valid once being integrated from its initial state, or what is the model valid prediction period (VPP)? To answer this question, uncertainty in ocean (or atmospheric) prediction should be investigated. It is widely recognized that the uncertainty can be traced back to three factors [*Lorenz*, 1969, 1984]: (1) measurement errors, (2) model errors such as discretization and uncertain model parameters, and (3) chaotic dynamics. The measurement errors cause uncertainty in initial and/or boundary conditions [e.g., *Jiang and Malanotte-Rizzoli*, 1999]. The discretization causes small-scale "subgrid" processes to be either discarded or parameterized. The chaotic dynamics may trigger a subsequent amplification of small errors through a complex response.

[3] The three factors cause prediction error. For example, an experiment on the Lorenz system was recently performed [*Chu*, 1999] through perturbing initial and boundary conditions by the same small relative error (10^{-4}). The vertical boundary condition error is transferred into the parameter error after turning the Saltzman model [*Saltzman*, 1962] into the Lorenz model [*Lorenz*, 1963] using a variable transform. The relative error is defined by the ratio between the rms error and rms of the three components. The Lorenz system has a growing period and an oscillation period. With the standard parameter values as used by *Lorenz* [1963], the growing period takes place as the nondimensional time from 0 to 22; and the oscillation period occurs as the nondimensional time from 22. During the growing period the relative error increases from 0 to an evident value larger than 1 for both initial and boundary uncertainties. During the oscillation period the relative error oscillates between two evident values: 4.5 and 0.1 for the initial uncertainty and 5.0 and 0.2 for the boundary uncertainty.

[4] Currently, some timescale (e.g., *e*-folding scale) is computed from the instantaneous error (defined as the difference between the prediction and reality) growth to represent the model VPP. Using instantaneous error (IE), model evaluation becomes stability analysis on small-amplitude errors in terms of either the leading (largest) Lyapunov exponent [e.g., *Lorenz*, 1969] or calculated from the leading singular vectors [e.g., *Farrell and Ioannou*, 1996a, 1996b]. The faster the IE grows, the shorter the *e*-folding scale is and, in turn, the shorter the VPP is.

[5] For finite amplitude IE, however, the linear stability analysis becomes invalid. The statistical analysis of the IE (both small-amplitude and finite amplitude) growth [*Ehrendorfer*, 1994a, 1994b; *Nicolis*, 1992], the information theoretical principles for the predictability power [*Schneider and Griffies*, 1999], and the ensembles for forecast skill identification [*Toth et al.*, 2001] become useful.

[6] The probabilistic properties of IE are described using the probability density function (PDF) satisfying the Liouville equation or the Fokker-Plank equation. *Nicolis* [1992] investigated the properties of the IE growth using a simple low-order model (projection of Lorenz system into most unstable manifold) with stochastic forcing. A large number of numerical experiments were performed to assess the relative importance of average and random elements in the IE growth.

[7] Although familiar and well understood, the IE growth rate is not the only factor to determine VPP. Other factors, such as the initial error and tolerance level of prediction error, should also be considered. The tolerance level of prediction error is defined as the maximum allowable forecast error. For the same IE growth rate the higher the tolerance level or the

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smaller the initial error, the longer the model VPP is. The lower the tolerance level or the larger the initial error, the shorter the model VPP is. Thus the model VPP is defined as the time period when the prediction error first exceeds a predetermined criterion (i.e., the tolerance level ε). The model predictability is then represented by a scalar VPP. The longer the VPP, the higher the model predictability is. In this study we develop a theoretical framework for model predictability evaluation using VPP and illustrate the usefulness and special features of VPP. The outline of this paper is depicted as follows. Description of prediction error of deterministic and stochastic models is given in section 2. Estimate of VPP is given in section 3. Determination of VPP for a one-dimensional stochastic dynamic system is discussed in section 4. The conclusions are presented in section 5.

2. Prediction Error

2.1. Dynamic Law

[8] Let $\mathbf{x}(t) = [x^{(1)}(t), x^{(2)}(t), \ldots, x^{(n)}(t)]$ be the full set of variables characterizing the dynamics of the ocean (or atmosphere) in a certain level of description. Let the dynamic law be given by

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t), \tag{1}$$

where $\boldsymbol{f}$ is a functional. Deterministic (oceanic or atmospheric) prediction is to find the solution of equation (1) with an initial condition

$$\mathbf{x}(t_0) = \mathbf{x}_0, \tag{2}$$

where $\mathbf{x}_0$ is an initial value of $\mathbf{x}$.

[9] With a linear stochastic forcing $q(t)\mathbf{x}$, equation (1) becomes

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) + q(t)(\mathbf{x}). \tag{3}$$

Here $q(t)$ is assumed to be a random variable with zero mean,

$$\langle q(t) \rangle = 0, \tag{4}$$

and pulse-type variance

$$< q(t)\, q(t') >= q^2 \delta(t - t'), \tag{5}$$

where the bracket < > is defined as the ensemble mean over realizations generated by the stochastic forcing, δ is the Delta function, and q^2 is the intensity of the stochastic forcing.

2.2. Model Error

[10] Let $\mathbf{y}(t) = [y^{(1)}(t), y^{(2)}(t), \ldots, y^{(n)}(t)]$ be the estimate of $\mathbf{x}(t)$ using the prediction model equation (1) or equation (3) with an initial condition

$$\mathbf{y}(t_0) = \mathbf{y}_0. \tag{6}$$

The prediction error vector is defined by

$$\mathbf{z} = \mathbf{x} - \mathbf{y},$$

at any time t ($>t_0$), and the initial error vector is defined by

$$\mathbf{z}_0 = \mathbf{x}_0 - \mathbf{y}_0.$$

If the components $[x^{(1)}(t), x^{(2)}(t), \ldots, x^{(n)}(t)]$ are not equally important in terms of prediction, the uncertainty of model prediction can be measured by the rms error

$$J(\mathbf{z}) = \langle \mathbf{z}'\mathbf{W}\mathbf{z} \rangle, \tag{7}$$

where $\mathbf{W}$ is the diagonal weight matrix, the superscript t denotes the transpose operator, and the bracket represents the ensemble average over realizations generated by stochastic forcing, uncertain initial conditions, and uncertain model parameters.

3. VPP

3.1. Indirect Estimate From IE Growth Rate

[11] Traditionally, the stability analysis is used to investigate the small-amplitude error dynamics due to the initial condition error,

$$\mathbf{z}(t_0) = \mathbf{z}_0, \qquad t = t_0, \tag{8}$$

with the IE growth rate and the corresponding e-folding timescale as the measures of the model predictability skill. It is reasonable to assume linear error dynamics for small-amplitude errors,

$$\mathbf{f}(\mathbf{z}, t) = \mathbf{A}(t)\mathbf{z}. \tag{9}$$

The first Lyapunov exponent is defined as

$$\lambda = \lim_{t\to\infty} \sup \frac{\ln(\|\mathbf{\Phi}(t, t_0)\|)}{t}, \tag{10}$$

where $\mathbf{\Phi}(t, t_0)$ is calculated by [*Coddington and Levinson*, 1995]

$$\mathbf{\Phi}(t, t_0) = \mathbf{I} + \int_{t_0}^{t} \mathbf{A}(s)ds + \int_{t_0}^{t} \mathbf{A}(r)dr \int_{t_0}^{r} \mathbf{A}(s)ds + \ldots,$$

where $\mathbf{I}$ is the unit matrix. Usually, the e-folding scale relating to the IE growth rate (or the Lyapunov exponent) is used to represent VPP.

3.2. Direct Calculation

[12] When state errors grow to finite amplitudes, the linear assumption equation (9) is no longer applicable, and the nonlinear effect should be considered. However, the prediction error cannot be less than the noise level (minimum limit) ξ_{noise} and greater than the tolerance level (maximum limit) ε. The ratio between the maximum and minimum limits is usually large:

$$z_1 = \varepsilon/\xi_{noise} >> 1, \tag{11}$$

in ocean models. Thus the rms error of prediction is bounded by the two limits

$$\xi_{noise}^2 \leq J(\mathbf{z}) \leq \varepsilon^2. \tag{12}$$

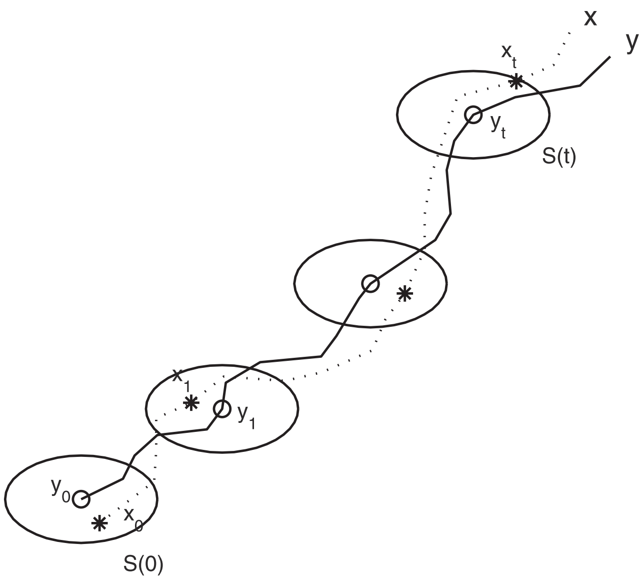

Figure 1. Phase space trajectories of model prediction **y** (solid curve) and reality **x** (dotted curve) and error ellipsoid $S_\varepsilon(t)$ centered at **y**. The positions of reality and prediction trajectories at time instances are denoted by asterisks and open circles, respectively. A valid prediction is represented by a time period $(t - t_0)$ at which the error first goes out of the ellipsoid $S_\varepsilon(t)$.

Two ellipsoids, S_ε and, S_ξ are defined by

$$J(\mathbf{z}) = \varepsilon^2 \qquad J(\mathbf{z}) = \xi^2_{\text{noise}},$$

with $\mathbf{y}(t)$ as the center (Figure 1).

[13] VPP, represented by a time period $(t - t_0)$ at which $\mathbf{z}$ (the error) goes out of the ellipsoid $S_\varepsilon(t)$, is a random variable, whose conditional PDF $P[(t - t_0)|\mathbf{z}_0]$ satisfies the backward Fokker-Planck equation [*Pontryagin et al.* 1962; *Gardiner*, 1983; *Ivanov et al.*, 1994]

$$\frac{\partial P}{\partial t} - [\mathbf{f}(\mathbf{z}_0, t)]\frac{\partial P}{\partial \mathbf{z}_0} - \frac{1}{2}q^2\mathbf{z}_0^2\frac{\partial^2 P}{\partial \mathbf{z}_0 \partial \mathbf{z}_0} = 0. \tag{13}$$

To solve equation (13), one initial condition and two boundary conditions (with respect to $\mathbf{z}_0$) are needed. Since the initial error $\mathbf{z}_0$ is always less than the given tolerance level (i.e., always inside the ellipsoid $S_\varepsilon(t)$), the conditional PDF of VPP at t_0 is given by

$$P[(0)|\mathbf{z}_0] = 1. \tag{14}$$

If the initial error $\mathbf{z}_0$ reache the tolerance level (i.e., $\mathbf{z}_0$ hits the boundary of $S_\varepsilon(t_0)$), the model loses prediction capability initially:

$$P[(t - t_0) \mid \mathbf{z}_0] = 0 \quad at \quad J(\mathbf{z}_0) = \varepsilon^2, \tag{15a}$$

which is the absorbing-type boundary condition. If the initial error reaches the noise level (i.e., $\mathbf{z}_0$ hits the boundary of $S_\varepsilon(t_0)$), the boundary condition becomes [*Gardiner*, 1983]

$$\frac{\partial P[(t - t_0) \mid \mathbf{z}_0]}{\partial z_0^{(j)}} = 0,\ J(\mathbf{z}_0) = \xi^2_{\text{noise}}, \tag{15b}$$

which is the reflecting boundary condition.

[14] The kth moment ($k = 1, 2, \ldots$) of VPP is calculated using PDF:

$$\tau_k\ (\mathbf{z}_0) = k\int_{t_0}^{\infty} P[(t - t_0) \mid \mathbf{z}_0] \quad (t - t_0)^{k-1}\, dt, \qquad k = 1, \ldots, \infty. \tag{16}$$

The mean and variance of VPP can be calculated from the first two moments

$$\langle\tau\rangle = \tau_1 \tag{17a}$$

$$\langle\delta\tau^2\rangle = \tau_2 - \tau_1^2, \tag{17b}$$

where the bracket denotes the average overrealizations.

3.3. Autonomous Dynamical System

[15] The predictability is usually time-dependent in ocean (or atmospheric) systems [*Toth et al.*, 2001]. Even for an autonomous dynamical system,

$$\mathbf{f} = \mathbf{f}(\mathbf{z}_0),$$

the PDF of VPP still varies with time (following the backward Fokker-Planck equation, equation (13)):

$$\frac{\partial P}{\partial t} - \mathbf{f}(\mathbf{z}_0)\frac{\partial P}{\partial \mathbf{z}_0} - \frac{1}{2}q^2\mathbf{z}_0^2\frac{\partial^2 P}{\partial \mathbf{z}_0 \partial \mathbf{z}_0} = 0.$$

We multiply this equation by $(t - t_0)$ and $(t - t_0)^2$, then integrate with respect to t from t_0 to ∞ and obtain the mean VPP equation

$$\mathbf{f}(\mathbf{z}_0)\frac{\partial \tau_1}{\partial \mathbf{z}_0} + \frac{q^2\mathbf{z}_0^2}{2}\frac{\partial^2 \tau_1}{\partial \mathbf{z}_0 \partial \mathbf{z}_0} = -1 \tag{18}$$

and the VPP variability equation

$$\mathbf{f}(\mathbf{z}_0)\frac{\partial \tau_2}{\partial \mathbf{z}_0} + \frac{q^2\mathbf{z}_0^2}{2}\frac{\partial^2 \tau_2}{\partial \mathbf{z}_0 \partial \mathbf{z}_0} = -2\tau_1. \tag{19}$$

Here the expression

$$\int_{t_0}^{\infty} P[(t - t_0) \mid \mathbf{z}_0]dt = 1$$

is used. Both equations (18) and (19) are linear, time-independent, and second-order differential equations with the initial error $\mathbf{z}_0$ as the only independent variable. Two boundary conditions for τ_1 and τ_2 can be derived from equations (15a) and (15b),

$$\tau_1 = 0, \qquad \tau_2 = 0, \qquad J(\mathbf{z}_0) = \varepsilon^2; \tag{20}$$

$$\frac{\partial \tau_1}{\partial \mathbf{z}_0} = 0, \qquad \frac{\partial \tau_2}{\partial \mathbf{z}_0} = 0, \qquad J(\mathbf{z}_0) = \xi^2_{\text{noise}}. \tag{21}$$

4. Example

4.1. One-Dimensional Stochastic Dynamical System

[16] We use a one-dimensional probabilistic error growth model [*Nicolis*, 1992]

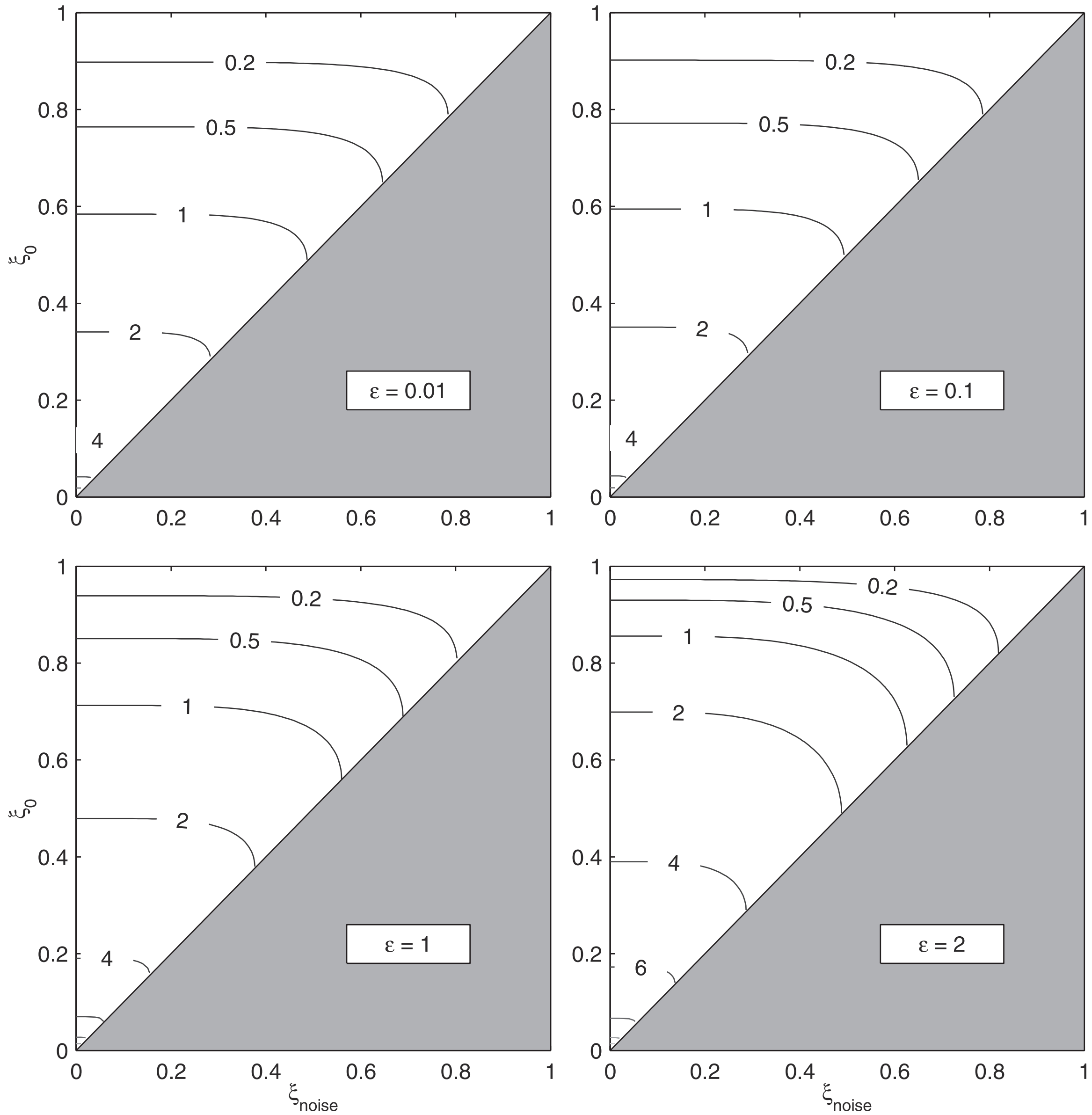

Figure 2. Contour plots of $\tau_1(\overline{\xi}_0, \overline{\xi}_{noise}, \varepsilon)$ versus $(\overline{\xi}_0, \overline{\xi}_{noise})$ for four different values of ε (0.01, 0.1, 1, and 2) using the *Nicolis* [1992] model with stochastic forcing q^2 = 0.2. The contour plot covers the half domain due to $\overline{\xi}_0 \geq \overline{\xi}_{noise}$.

$$\frac{d\xi}{dt} = (\sigma\xi - g\xi^2) + v(t)\xi, \qquad 0 \leq \xi < \infty, \tag{22}$$

as an example to illustrate the procedure in computing mean VPP and VPP variability. Here the variable ξ corresponds to the positive Lyapunov exponent σ, *g* is the nonnegative parameter whose properties depend on the underlying attractor, and $v(t)\xi$ is the stochastic forcing satisfying the condition

$$\langle v(t) \rangle = 0, \qquad \langle v(t)v(t') \rangle = q^2\delta(t - t').$$

Without the stochastic forcing $v(t)\xi$ the model (22) becomes the projection of the Lorenz attractor onto the unstable manifold.

4.2. Equations for the Mean and Variance of VPP

[17] How long is the model (22) valid once being integrated from the initial state? Or what are the mean and variance of VPP of equation (22)? To answer these questions, we should first find the equations depicting the mean and variance of VPP for equation (22). Applying the theory described in sections 3.2 and 3.3 to the model (22), the backward Fokker-Planck equation becomes

$$\frac{\partial P}{\partial t} - \left[\sigma\xi_0 - g\xi_0^2\right]\frac{\partial P}{\partial \xi_0} - \frac{1}{2}q^2\frac{\partial^2 P}{\partial \xi_0^2} = 0, \tag{23}$$

with the initial error ξ_0 bounded by

$$\xi_{noise} \leq \xi_0 \leq \varepsilon.$$

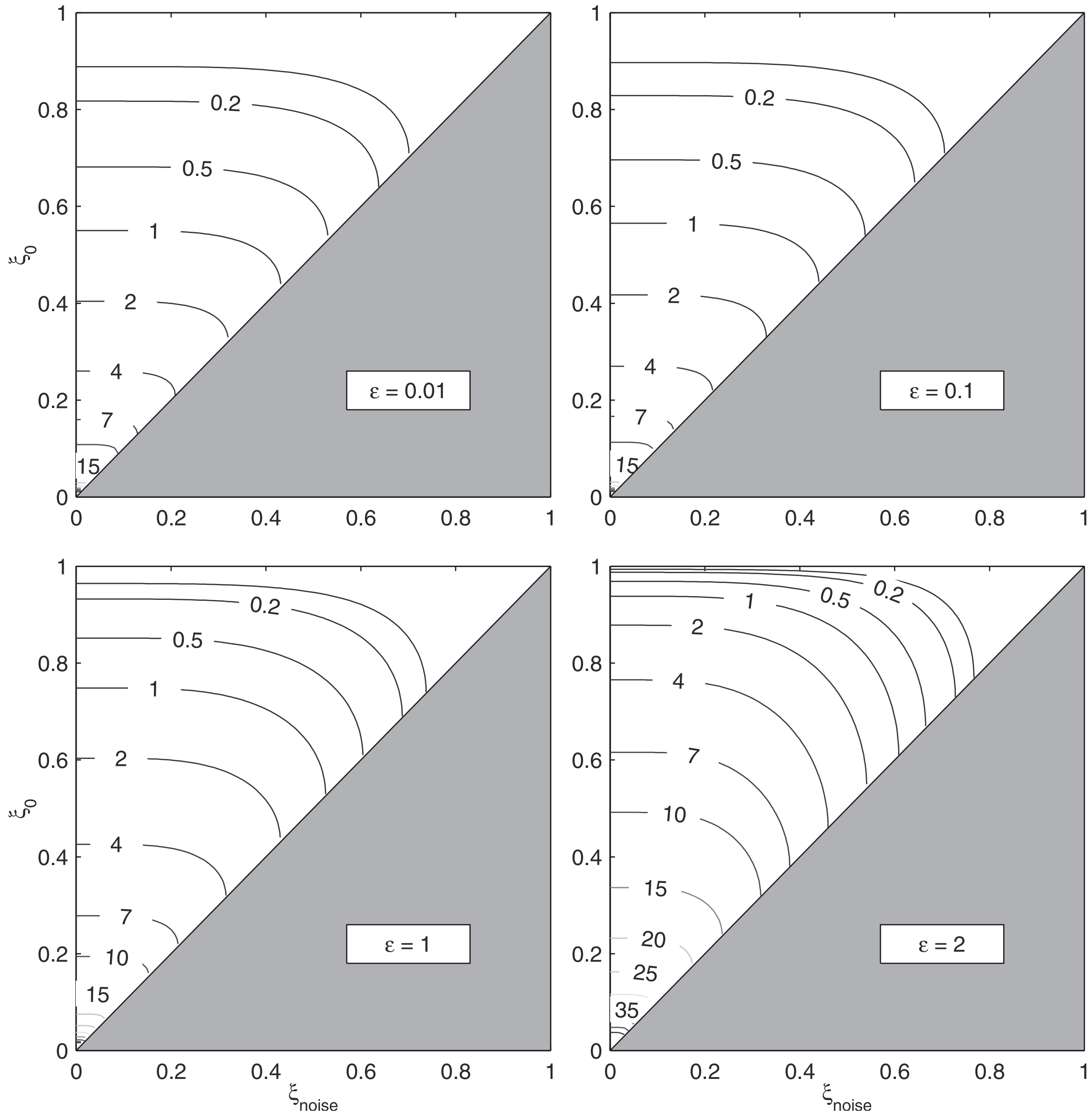

Figure 3. Contour plots of $\tau_2(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon)$ versus $(\bar{\xi}_0, \bar{\xi}_{noise})$ for four different values of ε (0.01, 0.1, 1, and 2) using the *Nicolis* [1992] model with stochastic forcing $q^2 = 0.2$. The contour plot covers the half domain due to $\bar{\xi}_0 \geq \bar{\xi}_{noise}$.

Furthermore, equations (18) and (19) become ordinary differential equations

$$(\sigma\xi_0 - g\xi_0^2)\frac{d\tau_1}{d\xi_0} + \frac{q^2\xi_0^2}{2}\frac{d^2\tau_1}{d\xi_0^2} = -1 \tag{24}$$

$$(\sigma\xi_0 - g\xi_0^2)\frac{d\tau_2}{d\xi_0} + \frac{q^2\xi_0^2}{2}\frac{d^2\tau_2}{d\xi_0^2} = -2\tau_1, \tag{25}$$

with the boundary conditions,

$$\tau_1 = 0; \qquad \tau_2 = 0, \ \xi_0 = \varepsilon; \tag{26}$$

$$\frac{d\tau_1}{d\xi_0} = 0, \qquad \frac{d\tau_2}{d\xi_0} = 0, \ \xi_0 = \xi_{noise}. \tag{27}$$

4.3. Analytical Solutions

[18] Analytical solutions of equations (24) and (25) with the boundary conditions (26) and (27) are

$$\tau_1(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon) = \frac{2}{q^2}\int_{\bar{\xi}_0}^{1} y^{-\frac{2\sigma}{q^2}}\exp\left(\frac{2\varepsilon g}{q^2}y\right) \cdot \left[\int_{\bar{\xi}_{noise}}^{y} x^{\frac{2\sigma}{q^2}-2}\exp\left(-\frac{2\varepsilon g}{q^2}x\right)dx\right]dy \tag{28}$$

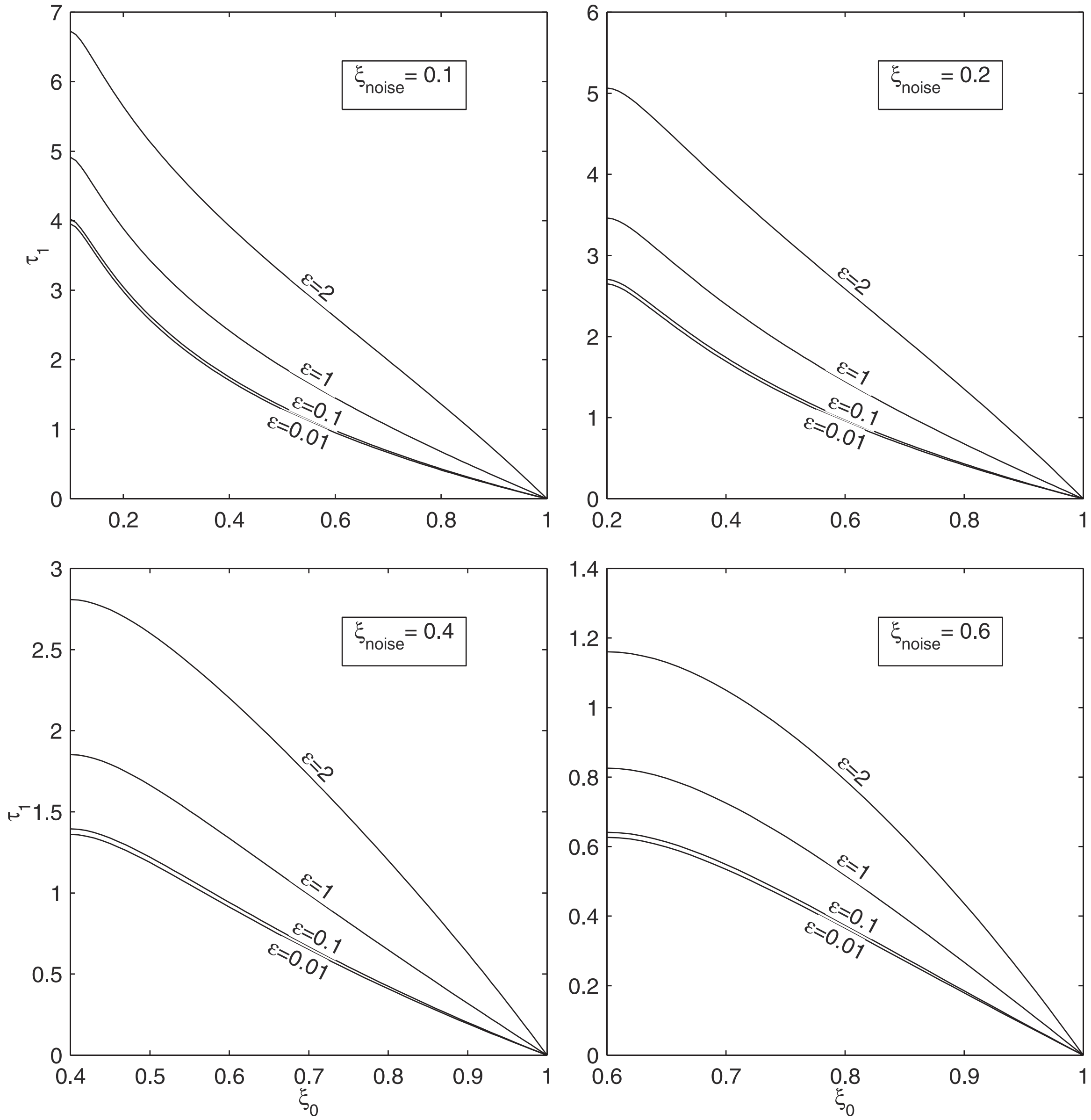

Figure 4. Dependence of $\tau_1(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon)$ on the initial condition error $\bar{\xi}_0$ for four different values of ε (0.01, 0.1, 1, and 2) and four different values of random noise $\bar{\xi}_{noise}$ (0.1, 0.2, 0.4, and 0.6) using the *Nicolis* [1992] model with stochastic forcing q^2 = 0.2.

$$\tau_2(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon) = \frac{4}{q^2} \int_{\bar{\xi}_0}^{1} y^{-\frac{2\sigma}{q^2}} \exp\left(\frac{2\varepsilon g}{q^2} y\right) \cdot \left[\int_{\bar{\xi}_{noise}}^{y} \tau_1(x) x^{\frac{2\sigma}{q^2}-2} \exp\left(-\frac{2\varepsilon g}{q^2} x\right) dx \right] dy \tag{29}$$

where $\bar{\xi}_0 = \xi_0/\varepsilon$ and $\bar{\xi}_{noise} = \xi_{noise}/\varepsilon$ are the nondimensional initial condition error and noise level scaled by the tolerance level ε, respectively. For given tolerance and noise levels (or user input) the mean and variance of VPP can be calculated using equations (28) and (29).

4.4. Dependence of τ_1 and τ_2 on $(\bar{\xi}_0, \bar{\xi}_{noise}/\varepsilon)$

[19] To investigate the sensitivity of τ_1 and τ_2 to $\bar{\xi}_0, \bar{\xi}_{noise}$ and ε, the same values are used for the parameters in the stochastic dynamical system (22) as were used by *Nicolis* [1992]:

$$\sigma = 0.64, \; g = 0.3, \; q^2 = 0.2. \tag{30}$$

Figures 2 and 3 show the contour plots of $\tau_1(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon)$ and $\tau_2(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon)$ versus $(\bar{\xi}_0, \bar{\xi}_{noise})$ for four different values of ε (0.01, 0.1, 1, and 2). The following features can be obtained: (1) For given values of $(\bar{\xi}_0, \bar{\xi}_{noise})$ (i.e., the same location in the contour plots) both τ_1 and τ_2 increase with

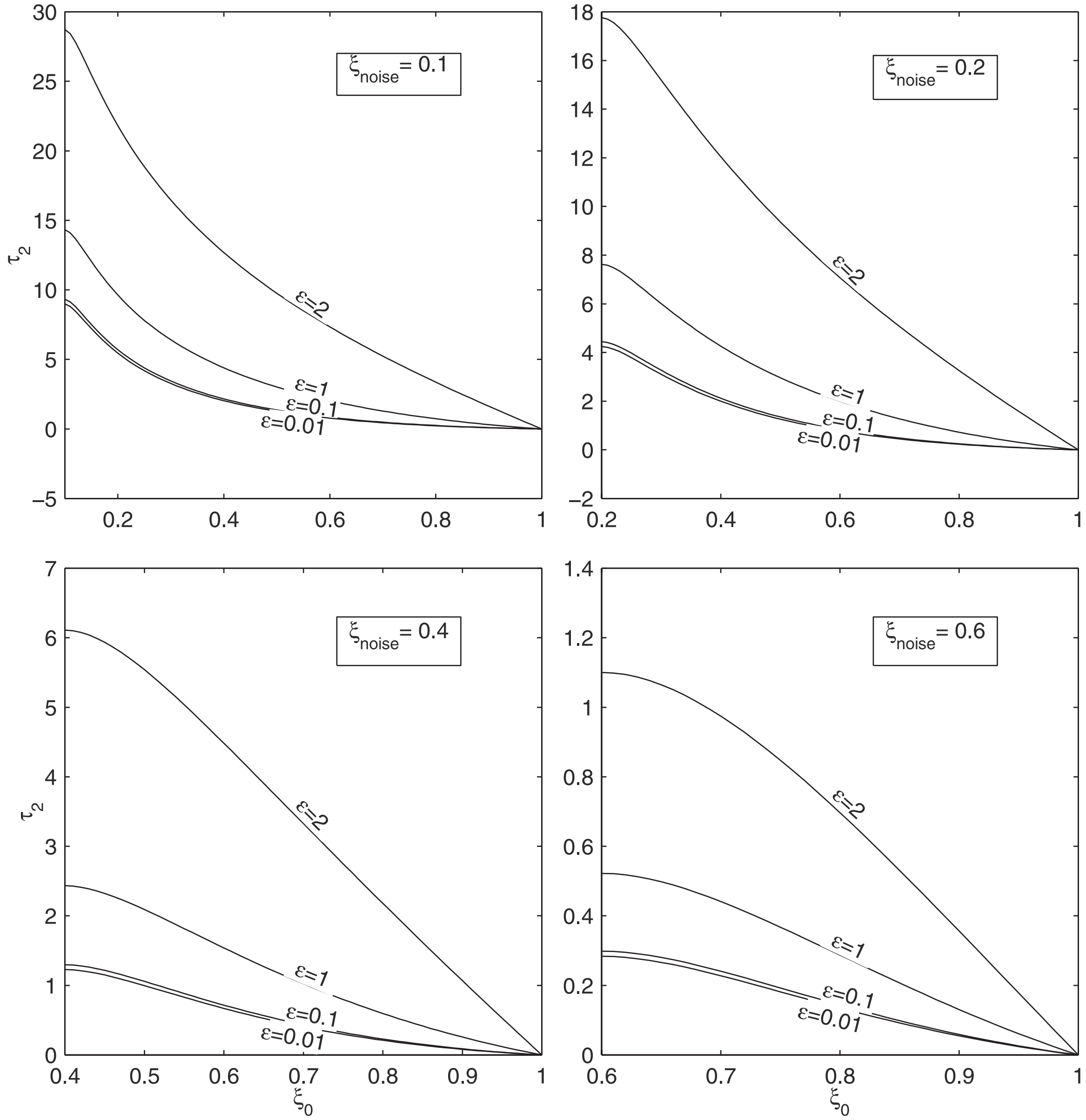

Figure 5. Dependence of $\tau_2(\overline{\xi}_0, \overline{\xi}_{noise}, \varepsilon)$ on the initial condition error $\overline{\xi}_0$ for four different values of ε (0.01, 0.1, 1, and 2) and four different values of random noise $\overline{\xi}_{noise}$ (0.1, 0.2, 0.4, and 0.6) using the *Nicolis* [1992] model with stochastic forcing q^2 = 0.2.

the tolerance level ε. (2) For a given value of tolerance level ε both τ_1 and τ_2 are almost independent on the noise level $\overline{\xi}_{noise}$ (contours are almost parallel to the horizontal axis) when the initial error $\overline{\xi}_0$ is much larger than the noise level $\overline{\xi}_{noise}$. This indicates that the effect of the noise level $\overline{\xi}_{noise}$ on τ_1 and τ_2 becomes evident only when the initial error $\overline{\xi}_0$ is close to the noise level $\overline{\xi}_{noise}$. (3) For given values of $(\varepsilon, \overline{\xi}_{noise})$ both τ_1 and τ_2 decrease with increasing initial error $\overline{\xi}_0$.

[20] Figures 4 and 5 show the curve plots of $\tau_1(\overline{\xi}_0, \overline{\xi}_{noise}, \varepsilon)$ and $\tau_2(\overline{\xi}_0, \overline{\xi}_{noise}, \varepsilon)$ versus $\overline{\xi}_0$ for four different values of tolerance level ε (0.01, 0.1, 1, and 2) and four different values of random noise $\overline{\xi}_{noise}$ (0.1, 0.2, 0.4, and 0.6). The following features are obtained: (1) τ_1 and τ_2 decrease with increasing $\overline{\xi}_0$, which implies that the higher the initial error, the lower the predictability (or VPP); (2) τ_1 and τ_2 decrease with increasing noise level $\overline{\xi}_{noise}$, which implies that the higher the noise level, the lower the predictability (or VPP); and (3) τ_1 and τ_2 increase with the increasing ε, which implies that the higher the tolerance level, the longer the VPP. Note that the results presented in this section are for a given value of stochastic forcing (q^2 = 0.2) only.

4.5. Dependence of τ_1 and τ_2 on Stochastic Forcing q^2

[21] To investigate the sensitivity of τ_1 and τ_2 to the strength of the stochastic forcing q^2, we use the same

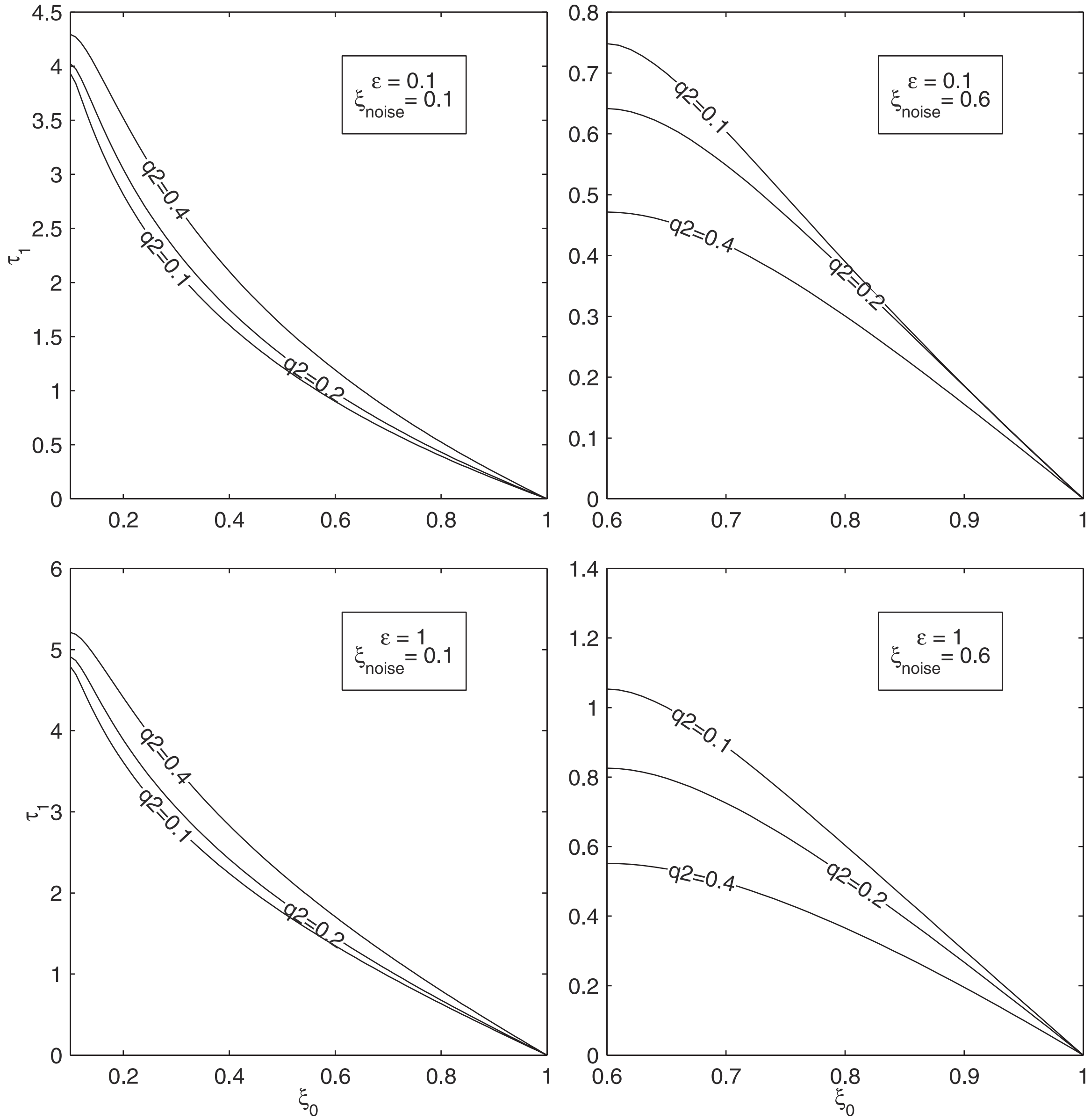

Figure 6. Dependence of $\tau_1(\overline{\xi}_0, \overline{\xi}_{noise}, q^2)$ on the initial condition error $\overline{\xi}_0$ for three different values of the stochastic forcing q^2 (0.1, 0.2, and 0.4) using Nicolis model with two different values of ε (0.1 and 1) and two different values of noise level $\overline{\xi}_{noise}$ (0.1 and 0.6).

values for the parameters (σ = 0.64 and g = 0.3) in equation (30) as were used by *Nicolis* [1992], except q^2, which takes values of 0.1, 0.2, and 0.4. Figures 6 and 7 show the curve plots of $\tau_1(\overline{\xi}_0, \overline{\xi}_{noise}, q^2)$ and $\tau_2(\overline{\xi}_0, \overline{\xi}_{noise}, q^2)$ versus $\overline{\xi}_0$ for two tolerance levels ε (0.1and 1), two noise levels $\overline{\xi}_{noise}$ (0.1 and 0.6), and three different values of q^2(0.1, 0.2, and 0.4) representing weak, normal, and strong stochastic forcing. Two regimes are found: (1) τ_1 and τ_2 decrease with increasing q^2 for large noise levels ($\overline{\xi}_{noise} = 0.6$), (2) τ_1 and τ_2 increase with increasing q^2 for small noise level ($\overline{\xi}_{noise} = 0.1$), and (3) both relationships (increase and decrease of τ_1 and τ_2 with increasing q^2 are independent of ε. These indicate the existence of stabilizing and destabilizing regimes of the dynamical system depending on stochastic forcing. For a small noise level the stochastic forcing stabilizes the dynamical system and increases the mean VPP. For a large noise level the stochastic forcing destabilizes the dynamical system and decreases the mean VPP.

[22] The two regimes can be identified analytically for small tolerance level ($\varepsilon \to 0$). The initial error $\overline{\xi}_0$ should also be small ($\overline{\xi}_0$ ε). The solution (28) becomes

$$\lim_{\varepsilon \to 0} \tau_1(\overline{\xi}_0, \overline{\xi}_{noise}, \varepsilon) = \frac{1}{\sigma - q^2/2} \left\{ \ln\left(\frac{1}{\overline{\xi}_0}\right) - \frac{q^2}{2\sigma - q^2} \overline{\xi}_{noise}^{\frac{2\sigma}{q^2}-1} \left[\left(\frac{1}{\overline{\xi}_0}\right)^{\frac{2\sigma}{q^2}-1} - 1 \right] \right\} \quad (31)$$

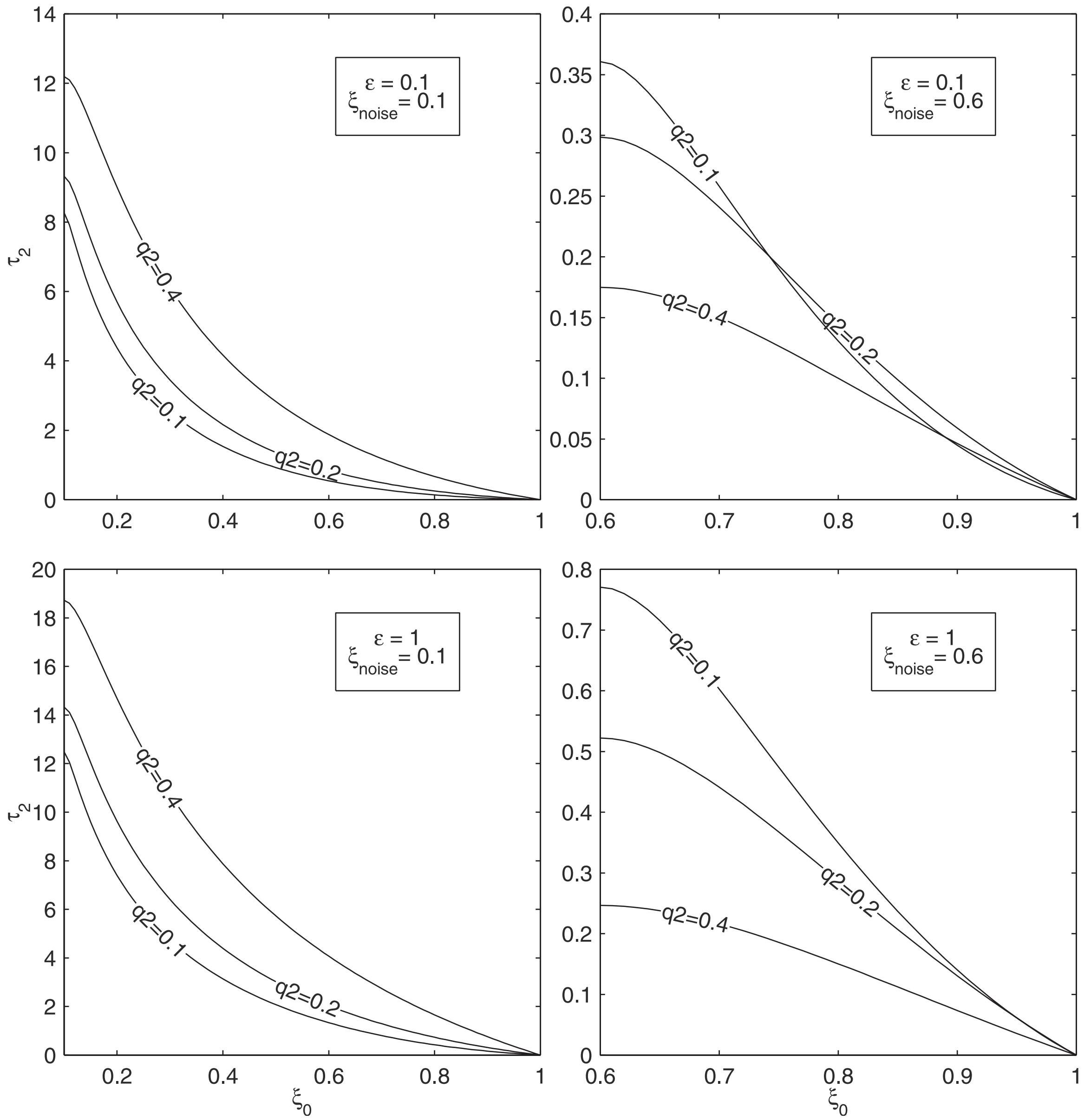

Figure 7. Dependence of $\tau_2(\bar{\xi}_0, \bar{\xi}_{\mathrm{noise}}, q^2)$ on the initial condition error $\bar{\xi}_0$ for three different values of the stochastic forcing q^2 (0.1, 0.2, and 0.4) using the *Nicolis* [1992] model with two different values of ε (0.1 and 1) and two different values of noise level $\bar{\xi}_{\mathrm{noise}}$ (0.1 and 0.6).

The Lyapunov exponent is identified as $\sigma - q^2/2$ for dynamical system (22) [*Has'minskii*, 1980]. For a small noise level ($\bar{\xi}_{\mathrm{noise}} = 1$) the second term in the bracket of the right-hand of equation (31),

$$R = -\frac{q^2}{2\sigma - q^2}\bar{\xi}_{\mathrm{noise}}^{\frac{2\sigma}{q^2}-1}\left[\left(\frac{1}{\bar{\xi}_0}\right)^{\frac{2\sigma}{q^2}-1} - 1\right], \qquad (32)$$

is negligible. The solution (31) becomes

$$\lim_{\varepsilon \to 0} \tau_1(\bar{\xi}_0, \bar{\xi}_{\mathrm{noise}}, \varepsilon) = \frac{1}{\sigma - q^2/2} \ln\left(\frac{1}{\bar{\xi}_0}\right), \qquad (33)$$

which shows that the stochastic forcing ($q \neq 0$) reduces the Lyapunov exponent ($\sigma - q^2/2$), stabilizes the dynamical system (22), and in turn, increases the mean VPP. On the other hand, the initial error $\bar{\xi}_0$ reduces the mean VPP.

[23] For a large noise level $\bar{\xi}_{\mathrm{noise}}$ the second term in the bracket of the right-hand side of equation (31) is not negligible. For a positive Lyapunov exponent, $2\sigma - q^2 > 0$, this term is always negative (see equation (32)). The absolute value of R increases with increasing q^2 (remember that $\bar{\xi}_{\mathrm{noise}} < 1$ and $\bar{\xi}_0 < 1$). Thus the term R destabilizes the one-dimensional stochastic dynamical system (22) and reduces the mean VPP.

5. Conclusions

1. The valid prediction period (a single scalar) represents the model predictability skill. It depends not only on

the instantaneous error growth but also on the noise level, the tolerance level, and the initial error. A theoretical framework was developed in this study to determine the mean (τ_1) and variability (τ_2) of the valid prediction period for a nonlinear stochastic dynamical system. The probability density function of the valid prediction period satisfies the backward Fokker-Planck equation. After solving this equation it is easy to obtain the ensemble mean and variance of the valid prediction period.

2. Uncertainty in ocean (or atmospheric) models is caused by measurement errors (initial and/or boundary condition errors), model discretization, and uncertain model parameters. This provides motivation for including stochastic forcing in ocean (atmospheric) models. The backward Fokker-Planck equation can be used for evaluation of ocean (or atmospheric) model predictability through calculating the mean valid prediction period.

3. For an autonomous dynamical system, time-independent second-order linear differential equations are derived for τ_1 and τ_2 with given boundary conditions. This is a well-posed problem, and the solutions are easily obtained.

4. For the *Nicolis* [1992] model the second-order ordinary differential equations of τ_1 and τ_2 have analytical solutions, which clearly show the following features: (1) decrease of τ_1 and τ_2 with increasing initial condition error (or with increasing random noise) and (2) increase of τ_1 and τ_2 with increasing tolerance level ε.

5. Both stabilizing and destabilizing regimes are found in the *Nicolis* [1992] model depending on the stochastic forcing. For a small noise level the stochastic forcing stabilizes the dynamical system and increases the mean VPP. For a large noise level the stochastic forcing destabilizes the dynamical system and decreases the mean VPP.

[24] **Acknowledgments.** This work was supported by the Office of Naval Research (ONR) Naval Ocean Modeling and Prediction (NOMP) Program and the Naval Oceanographic Office. Ivanov wishes to thank the National Research Council (NRC) for the associateship award.

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P. C. Chu, L. M. Ivanov, and C. Fan, Department of Oceanography, Naval Postgraduate School, 833 Dyer Road, Room 328, Monterey, CA 93943, USA. (chu@nps.navy.mil)