Air-ice-ocean feedback mechanisms, which are not conventionally incorporated within either climate or glacial models, are investigated to illustrate their potential role in generating ice advance/retreat on the time scale of $10^2$-$10^3$ years, i.e. for examining the internal causes for the ice oscillation.

Three main feedback loops are found from a coupled air-ice-ocean model developed in this paper: (a) ice advance – lower air temperature – ice freezing – ice advance; and (b) ice advance – higher ocean temperature – ice melting – ice retreat; (c) ice advance/retreat – modification of evaporation rate – change of ice accumulation rate and sea-level height – ice advance/retreat. The relative strength of the three feedback mechanisms determines the characteristics of the model: growing or decaying, oscillatory or non-oscillatory. The solutions show the generation of growing oscillatory modes with the time scale of $10^3$-$10^4$ years in certain parameter ranges.

**MODEL DESCRIPTION**

Ice advance/retreat

The total surface, ice-covered, and open ocean areas are set to be $1$, $N_i$, and $1 - N_i$, respectively (Fig. 1). The ice is assumed to be isotropic with thickness $H_i$. Following Weertman (1957), the ice-spreading rate is computed by:

$$\dot{H}_i = A \left( \frac{\rho_1}{\rho_g} H_i \right)^n$$

where $\rho_1$ is the ice density. $H_i$ is the height between the ice top and the sea-level (Fig. 1). Values of $n$ vary from about 1.5 to 4.2 with a mean of about 3, and for randomly oriented polycrystalline ice at $-10^\circ C$ and $n = 3$, a value $A = 3 \times 10^{-4}\text{a}^{-1}\text{KPa}^{-3}$ is reasonable for ice stress (Weertman, 1973). The angle brackets denote values averaged over ice thickness. The time rate of change of ice coverage $N_i$ contains mechanical and thermodynamical (ice edge freezing/melting) parts.

**Fig. 1. Physical processes in the coupled air-ice-ocean system.**

**Fig. 2. Feedback mechanisms among air, ice, and ocean.**
\[ \frac{\partial N_i}{\partial t} = N_i \langle \partial h \rangle + \frac{(1 - N_i) F}{H_i} \]  

where \( H_i \) is the ice thickness, and \( F \) is the freezing rate at the ice edge. Following Killworth (1979), the heat flux is accomplished through an exchange coefficient \( K \) and is of the form \( K(T - T_p) \), where \( T_p \) represents the freezing point of sea water. Therefore the freezing rate is computed by:

\[ F = - \frac{1}{\rho_i L_i} [\rho_w c_{pw} K_{wi}(T_w - T_p) + \rho_a c_{pa} K_{ai}(T_a - T_p)] \]

where \( K_{wi}, K_{ai} \) are the heat exchange coefficients for water-ice and air-ice, respectively, \( L_i \) is the latent heat of ice. If \( T_w, T_a \) are greater than \( T_p, F \) is taken as a negative value, which means ice melting.

**Time rate of change of ice thickness**

Based on the equation of continuity, Shumskiy (1965) proposed a method of determining ice-thickness changes by comparing the accumulation rate with horizontal strain-rates. The time rate of change of ice thickness becomes:

\[ \frac{\partial H_i}{\partial t} = c_s - H_i \langle \partial h \rangle - V_i \cdot \nabla H_i \]

where \( c_s \) is ice accumulation rate, and \( V_i \) is vertical mean ice velocity, which links to ice thickness and bedrock topography. It is commonly believed that the non-linear inertial term \( -V_i \cdot \nabla H_i \) in ice-sheet models relates to the long time scale of glacial cycles; however, for simplicity this term is neglected from the present model, i.e.:

\[ \frac{\partial H_i}{\partial t} = c_s - H_i \langle \partial h \rangle \]

which means that the current coupled system excludes the variations with long time scales comparable to the glacial cycles.

**Time rate of change of ocean depth**

From a thermodynamical point of view, the surface evaporation and ice freezing/melting are two major processes to change the ocean elevation, i.e.:

\[ \frac{\partial H_w}{\partial t} = -E - \frac{(1 - N_i) \rho_w F}{\rho_w} \]

where \( H_w \) is the depth of the ocean, \( \rho_w \) is the sea-water density, and \( E \) is the surface evaporation rate computed by the bulk formula:

\[ E = (1 - N_i) \gamma \frac{\rho_a}{\rho_w} \left[ q_s(T_w) - q_a \right] \]

where \( \gamma = (C_D \langle V_3 \rangle) \) is the moisture exchange coefficient. The relationships among the surface evaporation rate \( E \), the precipitation rate on the ocean \( P_{ir} \), and the ice accumulation rate \( c_s \) are simplified as:

\[ P_{ir} = (1 - N_i) E', \quad c_s = N_i E \]

which means the surface evaporation is the source for precipitation and ice accumulation. Following Killworth (1979), the heat fluxes are computed by:

\[ Q_{ai} = K_{ai}(T_a - T_i) \]

\[ Q_{wa} = K_{wa}(T_w - T_a) \]

\[ Q_{wi} = K_{wi}(T_w - T_i) \]

where \( c_{pa}, c_{pw} \) are the specific heats for air and ocean, respectively, and \( K_{ai} \), \( K_{wa} \), and \( K_{wi} \) are heat-exchange coefficients between the corresponding two components. The perturbation of lateral freezing rate is computed by:

\[ F' = \frac{(\rho_w c_{pw} K_{wi} T_w + \rho_a c_{pa} K_{ai} T_a)}{\rho_i L_i} \]

where \( L_i \) is the latent heat of the ice.
Timescales of perturbations

Four main timescales appear in this coupled system.

They are:

\[ \tau_a = \left[ \frac{K_{wi} + K_{wa}}{H_a N_i} \right]^{-1} \]  

(12)

which is the atmospheric relaxation timescale:

\[ \tau_w = \left[ \frac{K_{wi} + \frac{\rho_a \omega_a K_{wa}}{\rho_w c_{pw} H_w}}{H_w N_i} \right]^{-1} \]  

(13)

which is the oceanic relaxation timescale.

\[ \tau_e = \left( \frac{Q_{wa}}{Q_{wi}} - \frac{Q_{wa}}{Q_{wi}} \right) \frac{\tau_w}{\tau_w + \frac{\rho_a \omega_a \tau_w}{\rho_w c_{pw} H_w}} - \frac{E}{\rho_w c_{pw} H_w} \]  

(14)

which is the timescale for the change of evaporation rate due to the ocean warming/cooling induced by the ice advance/retrait, and:

\[ \tau_s = \left( \frac{\sigma}{\rho_i H_i} \right)^{-1} \]  

(15)

which is the ice-spreading timescale.

By using the values listed in Table I, these four timescales are estimated as:

\[ \tau_a \approx 1.6 \text{ year}, \ \tau_w \approx 66 \text{ year}, \ \tau_e \approx 4660 \text{ year} \]  

(16)

for \( H_i = 10 \text{ m} \), \( \tau_s = 3000 \text{ year} \)

where we assume that \( |Q_{wa}| > |Q_{wi}| \).

\section*{Table I. Model Parameters}

\begin{itemize}
  \item \( K_{wi} = 3 \times 10^2 \text{ m s}^{-1} \)
  \item \( K_{wa} = 10^2 \text{ m s}^{-1} \)
  \item \( K_{wi} = 4 \times 10^{-9} \text{ m s}^{-1} \)
  \item \( \gamma = 1.5 \times K_{wa} \) (Ghil, 1982)
  \item \( H_a = 5 \text{ km} \)
  \item \( H_w = 5 \text{ km} \)
  \item \( Q_{wa}/(\rho_a c_{pa} H_a) = 1 \text{ K s}^{-1} \)
  \item \( \sigma = 5, \ \omega = 3 \times 10^8 \text{ year}^{-1} \text{ kPa}^{-3} \)
\end{itemize}

Since the atmospheric and oceanic relaxation timescales \( \tau_a \) and \( \tau_w \) are so much shorter than the other two timescales \( \tau_s \) and \( \tau_e \), the atmospheric and oceanic temperature perturbations \( T_a \) and \( T_w \) almost "instantaneously" follow the ice advance/retrait processes, i.e.:

\[ T_a = A_3 T_s N_i, \ T_w = A_w T_w N_i \]  

(17)

where \( A_3 = (Q_{wa} + Q_{wi})/(\rho_a c_{pa} H_a) \), \( A_w = Q_{wa}/(\rho_w c_{pw} H_w) \).

Equation (17) clearly shows that the ice advance \((N_i > 0)\) causes the cooling of the atmosphere \((T_a < 0)\) and the warming of the ocean \((T_w > 0)\). Substitution of Equation (17) into Equation (11) leads to the connection of lateral freezing rate \( F' \) to \( N_i \):

\[ F' = -H_i (-a_d \omega_a \tau_w - a_d \omega_a \tau_w) N_i \]  

(18)

where \( a_d = \rho_a c_{pa} K_{wi} / (\rho_a c_{pa} H_a) \), \( \omega_a = \rho_w c_{pw} K_{wa} / (\rho_w c_{pw} H_w) \).

Stability parameters \( \sigma \) and \( \nu \)

Linearization of the basic Equations (2), (4), and (5) leads to:

\[ \frac{\partial N_i}{\partial t} = N_i \tau_s^{-1} \frac{(h_i - h_{w})}{H_i} - (1 - N_i) \tau_e^{-1} \sigma N_i \]  

(19)

\[ \frac{\partial h_{w}}{\partial t} = -H_i \tau_e^{-1} \left[ (1 - N_i) \omega_a \tau_w \right] \]  

(20)

where \( \sigma = \rho_a H_i / (\rho_w c_{pw} H_w) \), and \( \sigma \) is a non-dimensional parameter defined by:

\[ \sigma = \tau_e (a_d \omega_a \tau_w - a_d) \]  

(22)

For \( \sigma > 0 \), the ice has a "restoring force" induced by the ice advance/retrait. The mechanical feedback of the air and ocean into the ice can be easily noticed from the feedback of \( h_i \) and \( h_{w} \) change due to \( N_i \) into the ice advance/retrait through the first term in the right-hand side of Equation (19). This feedback largely depends on the parameter \( \nu = H_i / H_a \). The larger the value of \( \nu \), the stronger this feedback mechanism will be. Figure 3 shows the relationships among the model variables.

![Figure 3. Relationships among the model variables.](image)

The general solutions of Equations (19), (20) and (21) have the following forms:

\[ N_i(t) = \Sigma \beta_j e^{\omega_j t}, \ h_i(t) = \Sigma \beta_j e^{\omega_j t}, \ h_{w}(t) = \Sigma \beta_j e^{\omega_j t} \]  

(23)

where \( h_i, \ h_w \), and \( d_j \) \((j = 1,2,3)\) are the integral constants, and \( \omega_i, \ \omega_w \), and \( \omega_j \) are the eigenvalues which are the roots of the third-order algebraic equation:

\[ \omega^3 + a_2 \omega^2 + a_1 \omega + a_0 = 0 \]  

(24)

\section*{RESULTS}

We compute all roots of Equation (24) for different values of the parameters \( \sigma \) and \( \nu \), and obtain three roots at each pair of the parameters \((\sigma, \nu)\). One root among the

![Figure 4. Dependence of growth rate \( \omega_i \) on stability parameters \( \sigma \) and \( \nu \).](image)
three has a negative real part throughout the whole parameter space, representing the damping modes, in which we are not interested here. The other two roots, $\omega_1$ and $\omega_2$, have positive real parts somewhere in the parameter space, representing the existence of growing modes in certain parameter ranges. Figure 4 shows the dependence of growth rate ($10^{12} s^{-1}$) on $s$ for three different values of $v$. The horizontal axis $s$ changes from 0 to 10, representing the increase of thermodynamic stability. Oscillation takes place only in certain range of $s$ (Table II), i.e. for $s_H > s > s_c$, $\omega_1 \neq 0$. Out of this range, i.e. for $s < s_c$, or $s > s_H$, $\omega_1 = 0$, which represents non-oscillatory modes. Figure 5 shows the dependence of the period ($2\pi/\omega_1$) on $s$ for three different values of $v$.

![Figure 5. Dependence of period of oscillatory modes, $2\pi/|\omega_1|$, on stability parameters $s$ and $v$.](image)

**Table II. The Oscillation Intervals**

<table>
<thead>
<tr>
<th>$v$</th>
<th>$s_c$</th>
<th>$s_H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>3.75</td>
<td>5.75</td>
</tr>
<tr>
<td>100</td>
<td>3.75</td>
<td>5.00</td>
</tr>
<tr>
<td>200</td>
<td>4.00</td>
<td>4.75</td>
</tr>
</tbody>
</table>

The results are listed below:

1. The two parameters $s$ and $v$ play different roles in the coupled system. The negative feedback measured by $s$ stabilizes the system; however, the positive feedback measured by $v$ destabilizes the system. Because of the existence of positive feedback mechanisms, the coupled system is unstable unless the parameter $s$ exceeds some critical value $s_c$. Figure 4 indicates that $s_c = 4.25$.

2. Four different types of modes are generated due to the relative strength of positive to negative mechanism:

- $s < s_c$: non-oscillatory growing,
- $s_c < s < s_H$: oscillatory growing,
- $s_c < s < s_H$: oscillatory damping,
- $s > s_H$: non-oscillatory damping.

(3) The parameter $v$ serves as an instability parameter. In the growing regime (Fig. 4), the larger the value of $v$, the larger the growth rate will be. The mean growth rate of the growing modes varies from $2\times10^{-12} s^{-1}$. The doubling time during which the perturbation doubles its strength is:

$$T_d = \frac{\ln 2}{\omega_1} \approx 3660 \approx 10000 \text{ year}.$$

(4) The parameter $v$ also largely impacts on the periodicity of the coupled system. First, $v$ affects the critical values of $s_c$ and $s_H$. The larger the value of $v$, the smaller the range of oscillatory mode will be. The mean period varies from 2000–3000 years (for $v = 200$) to 4000–12000 years (for $v = 100$).

**CONCLUSION**

This conceptual air-ice-ocean coupled model shows a possible positive/negative feedback mechanism, induced by the hydrological cycle, among the atmosphere, ice, and ocean. The theory predicts the generation of oscillations in ice coverage, ice thickness, air and ocean temperatures, and sea-level height, on the time scale of $10^5-10^6$ years.

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