How Long Can a Model Predict?

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This question should be answered before any modeling effort

- How long is an ocean (or atmospheric) model valid once being integrated from its initial state?
Atmospheric & Oceanic Models

- \[ \frac{d \mathbf{X}}{dt} = f(\mathbf{X}, t) + q(t) \mathbf{X} \]
- Initial Condition: \( \mathbf{X}(t_0) = \mathbf{X}_0 \)
- Stochastic Forcing:
  - \( \langle q(t) \rangle = 0 \)
  - \( \langle q(t)q(t') \rangle = q^2 \delta(t-t') \)
Prediction Model

- \( Y \) --- Prediction of \( X \)

- Model: \( \frac{dY}{dt} = h(y, t) \)

- Initial Condition: \( Y(t_0) = Y_0 \)
Model Error

- \[ Z = X - Y \]
- Initial: \[ Z_0 = X_0 - Y_0 \]
Definition of Valid Prediction Period (VPP)

- VPP is defined as the time period when the prediction error first exceeds a predetermined criterion (i.e., the tolerance level $\varepsilon$).

- VPP is the First-Passage Time
Conditional Probability Density Function

- Initial Error: \( Z_0 \)
- \((t - t_0)\) \(\circ\) Random Variable
- Conditional PDF of \((t - t_0)\) with given \( Z_0 \circ\)
  - \( P[(t - t_0) | Z_0] \)
Two Approaches to Obtain PDF of VPP

- Practical
- Analytical (Backward Fokker-Planck Equation)
Gulf of Mexico Forecast System

- University of Colorado Version of POM
- 1/12° Resolution
- Real-Time SSH Data (TOPEX, ESA ERS-1/2) Assimilated
- Real Time SST Data (MCSST, NOAA AVHRR) Assimilated
- Six Months Four-Times Daily Data From July 9, 1998 for Verification
Model Generated Velocity Vectors at 50 m on 00:00 July 9, 1998
(Observational) Drifter Data at 50 m on 00:00 July 9, 1998
Reconstructed Drift Data at 50 m on 00:00 July 9, 1998 (Chu et al. 2002 a, b, JTECH)
Probability Density Function of VPP calculated with different tolerance levels

Non-Gaussian distribution with long tail toward large values of VPP (Long-term Predictability)
Power Law

\[ P(t_0, z_0, \varepsilon, t - t_0) \sim t^{-\gamma} \quad \text{for large } t. \]

Long-Term Predictability May Occur
Backward Fokker-Planck Equation (Analytical)

\[
\frac{\partial P}{\partial t} - [f(z_0, t)] \frac{\partial P}{\partial z_0} - \frac{1}{2} q^2 z_0^2 \frac{\partial^2 P}{\partial z_0 \partial z_0} = 0
\]
Moments

\[ \tau_1(z_0) = \int_{t_0}^{\infty} P(t_0, z_0, t-t_0)(t-t_0) \, dt \]

\[ \tau_2(z_0) = \int_{t_0}^{\infty} P(t_0, z_0, t-t_0)(t-t_0)^2 \, dt \]
Example: One Dimensional Error Model (Nicolis 1992)

- 1D Dynamical System (Maximum Growing Manifold of Lorenz System)

\[ \frac{d\xi}{dt} = (\sigma - g\xi^2) + \nu(t)\xi, \quad 0 \leq \xi < \infty \]

\[ \langle \nu(t) \rangle = 0, \quad \langle \nu(t)\nu(t') \rangle = q^2 \delta(t-t'). \]

\[ \sigma = 0.64, \quad g = 0.3, \quad q^2 = 0.2. \]
Mean and Variance of VPP

\[
(\sigma \xi_0 - g \xi_0^2) \frac{d \tau_1}{d \xi_0} + \frac{q^2 \xi_0^2}{2} \frac{d^2 \tau_1}{d \xi_0^2} = -1
\]

\[
(\sigma \xi_0 - g \xi_0^2) \frac{d \tau_2}{d \xi_0} + \frac{q^2 \xi_0^2}{2} \frac{d^2 \tau_2}{d \xi_0^2} = -2\tau_1
\]

\begin{align*}
\tau_1 &= 0, & \tau_2 &= 0 & \text{for } \xi_0 = \xi_0. \\
\frac{d \tau_1}{d \xi_0} &= 0, & \frac{d \tau_2}{d \xi_0} &= 0 & \text{for } \xi_0 = \xi_{\text{noise}}.
\end{align*}
Analytical Solutions

\[ \tau_1(\bar{\xi}_0, \bar{\xi}_{\text{noise}}, \varepsilon) = \frac{2}{q^2} \int_{\bar{\xi}_0}^{1} \frac{2\sigma}{q^2} y \exp\left(\frac{2\varepsilon g}{q^2} y\right) \left[ \int_{\bar{\xi}_{\text{noise}}}^{y} x^{2\sigma q^{-2}} \exp\left(-\frac{2\varepsilon g}{q^2} x\right) dx \right] dy \]

\[ \tau_2(\bar{\xi}_0, \bar{\xi}_{\text{noise}}, \varepsilon) = \frac{4}{q^2} \int_{\bar{\xi}_0}^{1} \frac{2\sigma}{q^2} y \exp\left(\frac{2\varepsilon g}{q^2} y\right) \left[ \int_{\bar{\xi}_{\text{noise}}}^{y} \tau_1(x) x^{2\sigma q^{-2}} \exp\left(-\frac{2\varepsilon g}{q^2} x\right) dx \right] dy \]

\[ \bar{\xi}_0 = \frac{\xi_0}{\varepsilon}, \quad \bar{\xi}_{\text{noise}} = \frac{\xi_{\text{noise}}}{\varepsilon} \]
Dependence of $\tau_1$ & $\tau_2$ on Initial Condition Error ($\frac{\xi_0}{\varepsilon}$)
Conclusions

- (1) VPP is an effective prediction skill measure (scalar).

- (2) Backward Fokker-Planck equation is a useful tool for predictability study.

- (3) Stochastic-Dynamic Modeling